2417

A second report on the teaching of arithmetic in schools

A report on the teaching of
arithmetic to the more able pupils
in secondary schools
prepared for

THE MATHEMATICAL ASSOCIATION

London: G. Bell & Sons Ltd

This book was taken from the Library of Extension Services Department on the date last stamped. It is returnable within 7 days.

6·30

A SECOND REPORT ON THE TEACHING OF ARITHMETIC IN SCHOOLS

Prepared for The Mathematical Association

A report on the teaching of arithmetic
to the more able pupils in secondary schools

INSTITUTE OF EDUCATION FOR WOMEN
2477
Dept. of Extension
SERVICE.
30.3.67
CALCUTTA-27

G. BELL & SONS, LTD
Portugal Street, London, W.C.2
1964

© The Mathematical Association
First published 1964

In 1958 the Teaching Committee agreed that the Report on *The teaching of arithmetic in schools* was in need of revision. A sub-committee was set up which reported to the Teaching Committee in March 1964. The revision necessary proved so extensive that the present report may be regarded as a new one. The members of this sub-committee were as follows:

MISS L. D. ADAMS
MR W. ARMISTEAD
MR J. K. BACKHOUSE (*Secretary* from 1960)
MISS H. BROMBY
MR C. T. DALTRY (*Secretary* to 1960)
MRS H. L. FUDGE
MR L. G. W. P. JONES
MR E. H. LOCKWOOD
MR C. G. NOBBS
MR J. S. PEEL (from 1960)
MR G. K. PRIESTMAN (to 1960)
MR A. W. RILEY (*Chairman of the Teaching Committee*)
MR T. R. THEAKSTON (*Secretary of the Teaching Committee*)
MR C. O. TUCKEY (*Chairman*)
MRS E. M. WILLIAMS

The printing of this Report follows the recommendations of British Standard 1991: Part 1: 1954, *Letter Symbols, Signs, and Abbreviations.*

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD
THE UNIVERSITY PRESS, GLASGOW

CONTENTS

CHAPTER PAGE

1. Introduction 1
2. New Developments in School Mathematics 3
3. Numbers 6
4. Some Remarks on the First Four Rules 13
5. Compound Units 15
6. Fractions and Decimals 20
7. Percentages 29
8. Ratio and Rate 33
9. Mensuration 44
10. Graphs 54
11. Squares and Square Roots 58
12. Logarithms 65
13. Approximations 74
14. Introduction to Statistics 79
15. Transition from the Primary School 83
16. Some Suggestions on Classroom Procedure 84

Appendix on Units 91

Index 93

INSTITUTE OF EDUCATION FOR WOMEN
Dept. of Extension
SERVICE.
★ CALCUTTA-27 ★

CHAPTER 1

INTRODUCTION

SINCE the publication of the last Arithmetic Report in 1932 there have been considerable changes in the background to the teaching of mathematics in schools.

Public opinion has become aware of the importance of mathematics in the life of the country and considerable thought has been given to methods of training mathematicians. Syllabuses have come under criticism, and much has been done towards the integration of the traditional branches of the subject.

The earlier report was written at a time when arithmetic was thought of as a separate school subject rather than as a part of elementary mathematics, when the use of tables of squares and square roots was rare, and when the farthing was still a coin to be reckoned with. Much attention was then given to computation and there was controversy among teachers as to the best methods, for example, in subtraction and in multiplication and division of decimals. Since then, opinion among teachers has hardened on some questions of alternative methods, which makes their discussion unnecessary. Contracted methods have been swept aside by the universal use of tables; and the whole subject is now viewed in better perspective as a part of elementary mathematics.

There has also been a change in the method of approach. In the early part of the century, as soon as new types of number such as fractions or decimals had been introduced, it was customary to teach the application to them of the 'four rules'—addition, subtraction, multiplication and division—for the sake of mastering the technique, somewhat apart from the possible use of this technique in cases likely to arise. But nowadays the view is that the problem comes first and the technique for solving it second. A new situation is presented as a challenge and time is allowed for pupils to attempt a solution. This is thought to be worth while no matter how fumbling their efforts may be.

This report discusses the teaching of arithmetic, as part of a general mathematics course, from the end of the primary school stage to the time when the General Certificate of Education is taken. It is written mainly for teachers of boys and girls of good ability such as are found in grammar schools and in the grammar school streams of comprehensive schools. There are many other pupils who are still not ready at 11-plus to begin such a course as is here envisaged, and appropriate guidance for their teaching may be found in the Association's Report, *Mathematics in secondary modern schools*.

In the present time of flux, it is not possible to anticipate what arithmetic will be included in school courses of the future. **So the inclusion of a particular example or topic in this Report must not be interpreted as a recommendation that these should necessarily be taught in schools.** Nor should the reader expect to find a comprehensive set of examples covering all the 'types' met with in arithmetic: a mathematical education does not consist of teaching pupils to tackle examination questions so much as to apply their knowledge of mathematics to situations both familiar and unfamiliar.

The time spent on arithmetic is nowadays likely to be shorter than it used to be and it is necessary to think of it always as part of elementary mathematics. But the importance of arithmetic remains. The science of number is a part of our culture and an ability to understand and use number is a fundamental social need. For many children there is a fascination in numbers and their properties; for all of them arithmetic is the foundation stone of mathematical knowledge. Its everyday applications are of interest to them and they like to feel they are learning something useful. But in fact they are doing more than this: they are beginning mathematics. Arithmetic leads on to algebra; particular numbers give way to variables, particular calculations to formulae, and the subject develops, as it did historically, through algebra and trigonometry to calculus and coordinate geometry. Through all of this, arithmetic is the pupil's sheet anchor, for by examining numerical cases he tests and clarifies his ideas and discovers his mistakes. The new topics which are now beginning to be taught in schools under the name of 'modern mathematics' have likewise their arithmetical side.

Finally it may be observed that arithmetic is the means by which most mathematics is applied, for the most general theory can only be brought to bear through the arithmetic of a particular case. In the world of applied mathematics numerical work is being pursued on an ever widening scale with the aid of machines and computers. This not only emphasises the importance of arithmetic but suggests that the lengthy calculations which were formerly undertaken in schools should now be omitted.

Arithmetic should be taught, from the start, with these ends in mind. This means, among other things, that the understanding of a process must be regarded as more necessary than the learning of a mechanical rule that produces the right answer; that in achieving this understanding accuracy and clear expression are vital; and that lengthy calculations should be avoided as likely to delay the grasp of underlying principles. It means too that a start on other branches of mathematics need not be unduly delayed; these give first-hand experience of some of the purposes of arithmetic besides being themselves the development to which arithmetic naturally leads.

CHAPTER 2

NEW DEVELOPMENTS IN SCHOOL MATHEMATICS

If arithmetic is the beginner's key to future progress, it should be taught in such a way that it leads on to the rest of mathematics, both elementary and advanced. This means not only algebra and trigonometry, coordinate geometry and calculus, but also the new topics now being introduced into school syllabuses—the binary scale, calculating machines, sets, groups, matrices, statistics, probability, and so on. As teachers we have to be aware of the far horizon but without losing sight of the ground under our pupils' feet. We must therefore take notice of the new developments and the experiments being made in widening the syllabus and should ask ourselves how arithmetic can prepare the way for this new work and take part in its introduction.

It may be thought that the function of arithmetic is to open the door rather than to begin the conduct of the tour, and certainly much can be done to provide the experience which will enable the new ideas to take root more easily. As an illustration we may consider the place of inequalities in elementary teaching, It is now realised that there has been too much emphasis in schools on equalities and not enough on inequalities. It has been said that mathematics begins with inequality, for counting and measuring are preceded by a recognition that some quantities are greater than others. A related notion of great importance is the idea of betweenness, and this has too often been neglected. For example, if a class is asked the value of π, there are sure to be some who say 'It is $3\frac{1}{7}$'. The answer 'π is between 3·14 and 3·15' is no more difficult and a much better preparation for later work. At all stages of the school work, examples can be found which use the idea of inequality and prepare the way for its more formal treatment later on. Some examples of this sort are:

1. In a rectangular park $\frac{1}{4}$ mile long and 200 yards wide there is a bandstand at the centre and the band is audible for a distance of 75 yards. Find the area of the park in which the band is inaudible.
2. A savage dog is tethered by a chain 12 ft long to a post 8 ft away from a straight path. For what distance along the path is one in danger?
3. Give some numbers whose square roots are between 5 and 6.
4. (*a*) If 144 is expressed as the product of two unequal factors, why is one less than 12 and the other greater than 12?

   (*b*) It is desired to test whether 149 is a prime number. If it is found to have no divisor less than 12, why is it certain that it is prime?

The comparison of fractions is perhaps the most obvious and natural occasion for the consideration of inequalities, and may be used for the introduction of the inequality signs. There may be different opinions as to when these signs should be introduced. Children adopt new symbols with enthusiasm and enjoy finding opportunities for their use. On the other hand, the important thing is that they should use inequalities readily in their thinking, and this may be encouraged more by using the phrases 'greater than', 'less than' and 'between' for some considerable time.

Another basic idea is the distinction between a proposition and its converse.

The symbol '$\therefore$' is often used by pupils in an indiscriminate manner and there are occasions when attention can be directed to its 'forward' direction. This can be emphasised by the use of the symbol $\Rightarrow$ (which should be read as 'implies' or 'only if') and the contrasting sign $\Leftrightarrow$ (read as 'implies and conversely' or 'if and only if').

Thus 'Area of square = 1089 sq. m
$\Leftrightarrow$side = 33 m';
but 'Rectangle measures 14 yd by 8 yd
$\Rightarrow$area = 112 sq. yd'.

For the 'new mathematics', as for the old, it is necessary for pupils to become accustomed to handling related sets of numbers. First there are sequences; and while these have commonly been treated as a part of algebra their beginning at least is arithmetical. Then there are related sequences such as illustrate functional dependence. Some suggestions will be found in this report for their introduction and use. Finally there are more elaborate two-dimensional patterns such as are found in statistical work. Results of a league competition provide an everyday example, Pascal's triangle a more purely mathematical one.

The suggestions made so far are aimed at preparing the ground for what may come later on. There is also the actual introduction, usually at 'Stage A' level, of some of the new topics. The important thing here is that the work should be relevant to the interests and capabilities of the pupils. It should be capable of being welded into the body of their mathematical knowledge and should be worth teaching for its own sake.

A good example is provided by the binary and other scales of notation. This has topical interest in connection with computers and is of value in helping pupils to understand the place-value system of numeration. It is also quite easy and can be taken at almost any stage. Calculating machines, too, are of interest in themselves and throw light on decimal multiplication and division. Somewhat later the slide rule can serve a similar purpose in increasing the understanding of logarithms.

Suggestions will be found in later sections of this report for early work on statistics, both graphical and computational. Again, choice and chance offer a wide field, too long neglected in elementary work. The forbidding names 'permutations', 'combinations' and 'probability' have obscured the fact that these subjects consist chiefly of counting, and that there are plenty of everyday examples much easier than most of those found in the algebra books.

Examples of sets occur in arithmetic, and the associated language and symbols can be introduced quite early. It is necessary to keep firmly in mind that the symbols express a meaning and should not be manipulated mechanically until much later. The teaching of such things as matrices and groups to young pupils is still in the experimental stage. These subjects can be brought down to 'Stage A' level, but the teaching must be careful and well-prepared, based always on simple examples.

An axiomatic approach to these new subjects is not in general suitable for pupils of school age. The teacher's object should be not to present a ready-made logical system but rather to introduce the pupils gradually to the processes of abstract thinking. It may be that, to professional mathematicians, what is taught in schools is an untidy, illogical, fragmented collection of bits and pieces, mostly physics, resting on no clearly defined basis. But, the nature of children and their learning being what it is, we know that we must proceed from the concrete to the abstract, from the particular to the general.

For instance it is natural and proper for children to begin thinking of a fraction as a broken-off bit of something. As their mathematical concepts are developed, many may appreciate that a fraction is an ordered pair of integers. In this country teachers would work towards this idea. They would never begin with it as a definition: that would be contradicting the whole trend of English mathematical education since the time of the dethronement of Euclid.

The foundations of mathematics are a difficult study and are best explored with a guide who knows when to leave well alone. Mathematics in its evolution has been concerned first with things, then with words and symbols, next with operations and relations, and so with logical structure. The symbols are abstractions and most symbols have more than one interpretation, according to the context. Thus '3' means 'three' (cardinal) and 'third' (ordinal). Some symbols have been casually handled in school teaching (e.g. '=' or '÷' or ':') and reasonable care should be given to defining the sense in which they are used. The 'minus' sign has two distinct meanings* which should be made clear at an early stage. In many other cases a more exact understanding can be developed as the pupil progresses. There also appears gradually the distinction between the exactness of logical consequence and the approximate nature of applications to the physical world. Thus a logical sense is developed and a foundation laid for axiomatic studies later on.

---

* See p. 8, paragraph numbered 4.

CHAPTER 3

NUMBERS

MATHEMATICS may be said to have two aspects: an outer one when it is applied to the practical affairs of the world, and an inner one when we are considering the relationships between the symbols or elements of the subject. By tradition, most of the arithmetic done in schools has been concerned with practical applications and with the acquiring of skill in computation. This section of the Report deals with work on the numbers themselves, and with the class of problems, of interest to many children from an early age, which serve to focus attention on these number properties and the structure of our number system.

In *The teaching of mathematics in primary schools*, stress was laid on the value of making friends with numbers and noticing the relationships between them. For example, patterns can be discovered by arranging the multiplication table as a square and the laws of numbers such as $a \times b = b \times a$ can be inferred. In the first year of the secondary school such work can provide a new approach to familiar routine calculation and enable a good deal of necessary revision to be done in a novel and interesting way. Work on numbers, accompanied perhaps by a historical approach to number and computation, can provide just that touch of novelty needed to revive interest.

There is a virtually unlimited amount of interesting material available but, if it is to retain its fascination, the teaching of it must be undertaken carefully. This work should be taken in small doses, alongside the usual work with techniques of computation and their application. There are great possibilities for teaching in the puzzle or problem manner. The class is set a problem as a challenge, e.g. '1, $1+3$, $1+3+5$, $1+3+5+7$ etc. are all perfect squares: why is this?' and then the different ways of approach are considered. Watching the pupils at this work will greatly assist the teacher to find out the children's interest and their capacity for dealing with relations between numbers.

Every teacher has his own interests in this field and we suggest that he should select according to his own taste and the response of the class. Problems, paradoxes, glimpses of the way ahead, and digressions into history and etymology all pay dividends by increasing the pupils' interest in the subject. Much useful material will be found in F. C. Boon's *Puzzle papers in arithmetic* and in other works listed at the end of this section. The list that follows is not exhaustive and, while many of these topics may find a place in the course, it is not intended that they should all be included. The reader is referred to Chapter 4 for work on the four rules.

Numerals. The different ways in which numbers have been written in Rome, Greece, Egypt and China, for instance, shed light on the advantages of our own system and serve to remind pupils that there have been civilizations other than our own.

The Roman numerals are important in that they are still in common use. Large numbers were cumbersome to write and the abacus was consequently an almost essential aid to calculating. The Alexandrian system at one time used in Greece had the advantage of writing numbers in a more compact form, but calculation was no more easy. Note that this system of numerals was based on the alphabet, whereas another system used in Egypt as early as 3500

B.C. was based on tallies. It is interesting to compare different systems, noting the number of symbols used, and bringing out the importance attached to place value and zero in our own system.

The growth of our number system. Teachers of mathematics should seize opportunities of developing in their pupils an understanding of our number system. When pupils arrive at secondary schools, they are usually familiar with the natural numbers 1, 2, 3, ..., with zero, and with fractions, but few of them have used directed numbers.

There are several points about which teachers should be clear in their own minds and these are discussed below. It is not intended that all of this should be taught in the early years of the secondary school but mention of the points in question is made in order that teachers should avoid misinforming or confusing their pupils about the subject: 'What the teacher knows and doesn't teach is more important than what he knows and does teach.'

1. There are various types of number in common use in arithmetic; terminology varies slightly and so we explain what we mean by them:

(i) the *natural numbers* are $1, 2, 3, \ldots$;

(ii) *fractions* are numbers in the form $\frac{a}{b}$, where a, b are natural numbers;

(iii) the *integers* are $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$;

(iv) *rational numbers* are numbers in the form $\frac{p}{q}$, where p, q are integers and and $q \neq 0$;

(v) *irrational numbers* are numbers, such as $\sqrt{2}$ and π, which cannot be expressed in the form $\frac{p}{q}$, where p, q are integers;

(vi) the phrase *directed numbers* is in common use and may refer to the integers or to the rational numbers.

Then there are other entities in mathematics which obey different sets of laws, for instance complex numbers, matrices, and vectors.

The various types of number have arisen historically as a result of problems in mathematics or in measurement. In the classroom, however, it is not necessary to teach the history of mathematics to see the need for the successive extension of the number system. For instance, if we restrict ourselves to the natural numbers, the operation of subtraction $a - b$ can only be performed if $a > b$. Then if we restrict ourselves to the integers, the operation of division $\frac{p}{q}$ can only be performed if p is divisible by q. Such exceptions to general rules are not pleasing and the wish for general statements leads to extensions of the system of numbers.

2. The same symbol is often used to denote logically distinct numbers, for example '3' may represent the natural number 3, the fraction $\frac{3}{1}$, the directed number $+3$, and even the complex number $3 + 0i$. There is, however, a (1, 1) correspondence between the corresponding elements of any pair of the ordered sets:

$$\{1, 2, 3, \ldots\},$$
$$\{+1, +2, +3, \ldots\},$$
$$\left\{\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \ldots\right\}$$

and in practice little difficulty is found in the ambiguous notation.

3. Logically, it is necessary to define the operations of addition and multiplication for the different numbers and then to examine whether the numbers so defined obey the usual 'laws of algebra'.

4. The symbols +, −, ×, ÷ are used in more than one way. For one thing, these operations are differently defined for the different numbers; this does not cause much misunderstanding, but the distinction between '−' as an order to subtract and '−' as the prefix of a negative number can and does cause confusion in the minds of some young pupils. For this reason some teachers use, at least for a time, a distinct notation for directed numbers, e.g.:

$$\ldots, (-3), (-2), (-1), 0, (+1), (+2), (+3), \ldots$$

$$\ldots, {}^-3, {}^-2, {}^-1, 0, {}^+1, {}^+2, {}^+3, \ldots$$

$$\ldots, \overset{-}{3}, \overset{-}{2}, \overset{-}{1}, 0, \overset{+}{1}, \overset{+}{2}, \overset{+}{3}, \ldots$$

Having raised what may appear to some readers to be unnecessary complications, we shall try to give some indication of how these matters may be dealt with in the classroom.

1. The existence of different types of numbers will be obvious to pupils as their mathematical education proceeds. Any reluctance to accept a new type of number because of its strangeness may be reduced by some form of visual representation: for fractions, by shaded parts of a whole; for directed numbers by points on a scale; and for irrational numbers such as $\sqrt{2}$ and π, by the length of a line. It also helps to remind the pupils of previous extensions to the number system. The better classes may be given the *reductio ad absurdum* proof that $\sqrt{2}$ is irrational.

2. The use of one numeral to denote different numbers does not usually cause lasting trouble. However, it is worth remarking that a pupil who writes $\frac{5}{6}+\frac{7}{6}=\frac{12}{6}=\frac{2}{1}$ is logically more consistent than one who gives the answer as 2. A common misunderstanding about complex numbers is expressed by saying that i is the square root of the directed number -1. In fact, there is no square root of the directed number -1, but the square root of the complex number $-1+0i$ is $0+i$.

3. A consideration of the introduction of addition and multiplication of fractions has been given in Chapter 6. Directed numbers are often considered to be part of an algebra course but some suggestions about their treatment are given below. For a fuller treatment of the subject, see Appendix A2 of the Association's Report, *The teaching of algebra in schools*.

The existence of directed numbers is known to pupils entering secondary schools from the weather forecasts which now give temperatures in Centigrade. It is necessary, however, to make sure that the pupils understand that the origin and the positive direction may be chosen arbitrarily and that the negative is the opposite direction.

When discussing how to add directed numbers, the teacher should be clear in his or her own mind that the law is a matter of definition but will probably not wish to put it quite so bluntly to the class. The definition chosen is not an arbitrary one; rather, the addition and multiplication of directed numbers are chosen to fit in with the addition and multiplication of natural numbers. This is an example of the process of generalisation, which is one of the characteristic features of mathematics. We choose the rules of composition so that

$${}^+a + {}^+b = {}^+(a+b) \qquad \text{(directed numbers)}$$

corresponds to

$$a+b=(a+b) \qquad \text{(natural numbers)}$$

and so that ${}^+a \times {}^+b = {}^+ab$ corresponds to $a \times b = ab$.

In the classroom, addition and subtraction with natural numbers can be represented by movements in opposite directions on a number scale. So addition and subtraction of directed numbers are represented by similar movements. Thus we can interpret $^+3 + {}^-2$ by some such action as, 'Move to $^+3$ on the scale, face the "+" direction and move $^-2$ places (i.e. 2 places in the opposite direction). We arrive at $^+1$.' So we say that $^+3 + {}^-2 = {}^+1$. Again, for $^+3 - {}^-2$ we can say, 'Move to $^+3$ on the scale, face the "−" direction and move $^-2$ places (i.e. 2 places in the opposite direction). We arrive at $^+5$.' So we say that $^+3 - {}^-2 = {}^+5$.

An effective way of introducing multiplication of directed numbers is to write up the beginnings of a multiplication table:

	$^-3$	$^-2$	$^-1$	0	$^+1$	$^+2$	$^+3$
$^+3$				0	$^+3$	$^+6$	$^+9$
$^+2$				0	$^+2$	$^+4$	$^+6$
$^+1$				0	$^+1$	$^+2$	$^+3$
0				0	0	0	0
$^-1$							
$^-2$							
$^-3$							

The class usually suggests that the product of a positive and a negative number should be negative, and then the sequences in the rows and columns bring most of the pupils to see that the product of two negative numbers should be taken to be positive. To some a more convincing method is to choose some application which is familiar to the class and which makes sense with negative numbers, for example, for constant speed,

$$\text{distance} = \text{speed} \times \text{time};$$

or, for a clock gaining or losing at a constant rate,

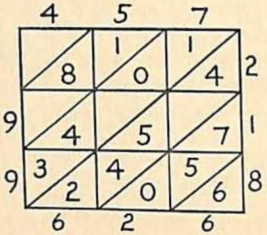
$$\text{gain} = \text{rate} \times \text{time}.$$

The rules of multiplication of directed numbers can then be chosen to fit in with the physical interpretation.

It should be emphasised that operations on directed numbers should only be introduced *when the need for them arises*. This means that multiplication and division may well be considered much later than addition and subtraction unless, of course, a pupil raises the question of how these operations are carried out. Time spent on clarifying the pupils' ideas on the number system is time well spent. It will ease the treatment later on of surds and complex numbers; but above all it will help to show the ordinary pupil that the rules of mathematics are reasonable, and not arbitrary.

Methods of calculation. Our present methods of long multiplication and division were developed long after the invention of our number system. Their use depends on the ability to add and subtract, and on a knowledge of multiplication tables up to 10 × 10. These were probably not commonly memorised in this country, even by educated people, until the sixteenth century. Consequently a number of other methods have been employed.

The abacus dispenses with the need for a knowledge of any number relationships and, in the hands of a skilled calculator is as fast as our written methods. The 'grating' method of multiplication was used in many countries, including India and China. It is of interest because of the neat diagonal device for moving on the 'tens' figure in each of the partial products. Here it is shown for 457 × 218 = 99 626.



Before the adoption of our present method of long division, which was not in general use in England until the beginning of the eighteenth century, the 'galley' or 'scratch' method was common. 384 is divided by 13 below. The finished calculation would appear as in (vi), but the steps are shown with explanations below each stage to show what has just been done.

$$\begin{array}{ccc|c} 3&8&4&\\ 1&3&& \end{array} \qquad \begin{array}{ccc|c} 1&&&\\ \cancel{3}&8&4&2\\ \cancel{1}&3&& \end{array} \qquad \begin{array}{ccc|c} 1&2&&\\ \cancel{3}&\cancel{8}&4&2\\ \cancel{1}&\cancel{3}&& \end{array}$$

(i) (ii) (iii)

Subtract 10 × 20. Subtract 3 × 20.

$$\begin{array}{ccc|c} 1&2&&\\ \cancel{3}&\cancel{8}&4&2\\ \cancel{1}&\cancel{3}&3&\\ &1&& \end{array} \qquad \begin{array}{ccc|c} &3&&\\ \cancel{1}&\cancel{2}&&\\ \cancel{3}&\cancel{8}&4&2\;9\\ \cancel{1}&\cancel{3}&3&\\ &\cancel{1}&& \end{array} \qquad \begin{array}{ccc|c} &\cancel{3}&&\\ \cancel{1}&\cancel{2}&7&\\ \cancel{3}&\cancel{8}&\cancel{4}&2\;9\\ \cancel{1}&\cancel{3}&\cancel{3}&\\ &\cancel{1}&& \end{array}$$

(iv) (v) (vi)

Write down divisor again. Subtract 10 × 9. Subtract 3 × 9.

The quotient is 29 and the remainder is 7.

The inventions of decimal and common fractions, and of logarithms, deserve some attention. Books on the history of mathematics give details of these and many other aspects of the history of calculating.

Some teachers use calculating machines to give a fresh approach to number operations and the decimal notation.

Scales of notation. A study of scales of notation, besides being of interest to many pupils, provides an opportunity to develop in them a greater insight into the way in which we usually write our numbers; this is especially important when they are starting on decimals. By the secondary stage, pupils are familiar with numbers to the base ten and they have a store of associations with written numbers up to about 100. Thus 39 is an odd number, it is equal to 3 × 13, and so on. These associations will not be unduly upset by experiments in using figures to write numbers to other bases. Thus 42 in decimal notation is 4 tens and 2, or 6 × 7, and 42 pence is 3s. 6d. But the digits 4 and 2 in 42 when the base is *five* indicate 4 fives and 2, i.e. twenty-two in decimal notation. Again, in the table of threes, for example,

	Scale of ten	*Scale of three*
1 three =	3	10
2 threes =	6	20
3 threes =	9	100
4 threes =	12	110
5 threes =	15	120
6 threes =	18	200
7 threes =	21	210
8 threes =	24	220
9 threes =	27	1000

It will be noted that the scale of three involves only three digits, 0, 1 and 2.

The use of ten fingers for the purpose of tallying led naturally to the use of a decimal system of notation. Had the human race generally used the twelve finger joints of one hand, thumb excluded, for the same purpose, we should have the duodecimal system instead. The history of mathematics provides other examples, but the advent of electronic calculating machines has brought the scale of two into prominence. Since only the symbols 1 and 0 are needed to represent numbers, machines can be made to represent a number by a set of electric currents which either do or do not flow. The scale of two has other well-known applications. In kitchen scales, the commonly used weights are $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 4, 8 oz and 1, 2 lb. (It is an interesting digression to consider what weights are needed if they may be placed on both sides of the scales.) 'Peasant' multiplication, age-telling cards and the game of Nim all make use of the scale of two.

Prime numbers and prime factors. The expressing of numbers in prime factors has an interest of its own, besides being a useful preliminary to factor work in algebra. It leads to the discovery of the sequence of prime numbers and their curiously irregular distribution, and it also provides occasion for the introduction of indices and allows the index laws to be discovered through simple examples. Work on prime factors should not involve large numbers but one of its aims should be to help the pupils to become familiar with the factors of the smaller numbers, say less than 100, together with squares, cubes and numbers such as 112 which frequently occur in calculations. Tests of divisibility arise naturally from this work and can be discussed in the course of it; treated this way they will be of much greater interest than if they are taught formally. Square roots, and possibly cube roots, can be found by prime factors at this stage and, while an introduction to fractional indices may not be appropriate so early in the course, it is likely that some pupils will ask what happens when the indices are not exactly divisible by two or three respectively, e.g.:

$$\sqrt{5^3}, \quad \sqrt[3]{2^4},$$

and a seed may be sown which will prepare them for later work.

Other topics. Easy sequences, the integers, squares, cubes and triangular numbers all have interesting applications and are readily linked with algebra through the finding of a formula for the *n*th term. A little work on recurring decimals is usually called for by the pupils' questions. The concept of number needs to be extended for later work, although this may be more suitable for the algebraic side of the mathematics course.

The L.C.M. and H.C.F. are only worth doing if they are properly understood, and of the two, the Least Common Multiple is the more important. It gives precision to the idea of the common denominator when fractions are added, and may well be introduced as the Least Common Denominator. This is particularly useful in algebra when the denominators have common factors. First the teacher must make sure that all the pupils understand the meaning of the word 'multiple'. From there, the next step is to find multiples which are common (another word which will not be understood by everyone) to two or more numbers, and then the pupils may find some least common multiples. As with most parts of arithmetic, the emphasis in teaching needs to be on the underlying ideas of the work; the way in which the work is to be set out will vary from child to child, with the teacher or other pupils suggesting improvements. The Highest Common Factor, if undertaken, can readily be treated in a similar manner—not immediately afterwards, but only when the method of finding the L.C.M. has been mastered.

For those teachers who wish to learn more about such topics as have been mentioned in this chapter, a list of books is included.

Ball, W. W. R., and Coxeter, H. S. M., *Mathematical recreations and essays.* (Macmillan)
Boon, F. C., *A companion to school mathematics.* (Longmans)
Boon, F. C., *Puzzle papers in arithmetic.* (Bell)
Bowman, M. E., *Romance in arithmetic.* (U.L.P.)
Dantzig, T., *Number, the language of science.* (Allen and Unwin)
Heafford, P., *Mathematics for fun.* (Hutchinson)
Hooper, A., *Makers of mathematics.* (Faber)
Kraitchik, M., *Mathematical recreations.* (Allen and Unwin)
Land, F., *The language of mathematics.* (Murray)
McKay, H., *Odd numbers.* (O.P.)
Reid, C., *From zero to infinity: What makes numbers interesting.* (Routledge & Kegan Paul)
Sanford, V., *A short history of mathematics.* (Harrap)
Shackle, G. L. S., *Mathematics by the fireside.* (O.P.)
Smith, D. E., *History of mathematics:* Vol. I, *General survey of the history of elementary mathematics.*
Vol. II, *Special topics of elementary mathematics.* (Dover)
Smith, D. E., *Source book in mathematics*, 2 vols. (Dover)
Smith, D. E., *Number stories of long ago.* (Ginn)
Smeltzer, D., *Man and number.* (A. & C. Black)
Turnbull, H. W., *The great mathematicians.* (Methuen) (Also E.U.P. paperback)

CHAPTER 4

SOME REMARKS ON THE FIRST FOUR RULES

AT first a child's main interest in the first four rules lies in what he can do with them. He is chiefly concerned with getting the answer. By the time he has reached the secondary school it ought to be possible to divert some interest to the operations themselves and this can be valuable preparation for later work. We might begin by noticing that, in the case of addition and multiplication, the order in which the numbers are taken does not influence the answer. Then we ought to recognise that the four operations are not independent, but subtraction is another aspect of addition and division another aspect of multiplication. Questions such as the following will lead up to these points: What number must be added to 72 to give 95? By what number must 8 be multiplied to give 48? A number is divided by 13 and the answer is 15; what is the number? Practice in the correct reading and writing of particular number statements can be brought in here:

$3+5=8$,	'3 plus 5 equals 8';
$8-3=5$,	'8 minus 3 equals 5';
$5\times 3=15$,	'5 multiplied by 3 equals 15';
$15\div 3=5$, or $\frac{15}{3}=5$,	'15 divided by 3 equals 5'.

Thus we reach some understanding of two fundamental truths:

(i) Subtraction answers the question 'What must be added to A to give B?'
(ii) Division answers the question 'By what must A be multiplied to give B?'

This is useful preparation for later work. For example,

(i) If $3x=11$, then $x=\frac{11}{3}$;
(ii) If $x+3=8$, then $x=8-3$;
(iii) From $(-5)+(+11)=(+6)$ we can deduce
$$(+6)-(-5)=(+11)$$
and hence the rule that $-(-5)$ is equivalent to $+(+5)$.

When we come to velocity, the statement

'(the velocity of an aircraft relative to the earth)
=(velocity of the aircraft relative to the air)
+(velocity of the air relative to the earth)'

may be re-written

'(the velocity of an aircraft relative to the air)
=(velocity of the aircraft relative to the earth)
−(velocity of the air relative to the earth)'

and this leads to the general rule for finding the velocity of A relative to B.

It is interesting to note that the addition aspect of subtraction is the basis of what is probably the commonest use of subtraction in everyday life, the reckoning of change. In finding what change to give from a 10s. note after making a purchase costing 2s. 8½d., the process always is '2s. 8½d. and 3½d. makes 3s., and 7s. makes 10s.'

The fundamental processes are bound to come up for review from time to time in the early stages at the secondary school. In general it is *not* advisable to try to alter methods of computation which are thoroughly known. This may mean that several methods, particularly in subtraction and multiplication, are in use in a class, and occasionally the teacher has to remember to consider separately the needs of the children using alternative methods. This inconvenience is preferable to compelling children to unlearn something that has already become mechanical. Such consideration of fundamental techniques as is necessary should aim at clarifying principles. For example the fairly common experience of finding a pupil dividing by 1000 by long division should be made an excuse for a further discussion on place value. Other examples bearing on the same principle are multiplications like 3742 × 1003 where many pupils, left to themselves, would write down two rows of four noughts.

The method of long division is recommended when it is desired to find a remainder. Here again it may be worthwhile looking at the mechanical process in a different way in order to achieve greater understanding. For example

$$\begin{array}{rlrl}
 & 14 & & \\
13\,) & \overline{186} & & \\
 & \underline{13} \ \dots\dots\dots & 130 = & 10 \times 13 \\
 & 56 & & \\
 & \underline{52} \ \dots\dots\dots & 52 = & 4 \times 13 \\
 & 4 \ \dots\dots\dots & 4 = & \underline{\qquad\quad 4} \\
 & & 186 = & 14 \times 13 + 4
\end{array}$$

Division by factors can be useful when only the quotient is required.

The aim throughout this early work should be the progressive introduction of mathematical notation, and usage. For example, the statement of an answer to a division in the form of quotient and remainder should give way, in suitable cases, to an answer expressed as a mixed number (with or without decimals). Moreover, the pupil should gradually be weaned from his natural habit of working out each stage of a calculation before proceeding to the next. The aim to set before him is the formulation of the answer as a mathematical statement before the calculation is begun, and it can be pointed out that by doing so the work can often be shortened.

CHAPTER 5

COMPOUND UNITS

In the application of numbers to measurement, the basic idea is to define a unit and count how many of them are needed to make up the quantity to be measured. When the need for greater precision is felt there are two main lines of advance:

1. by the use of a set of related units such as yards, feet and inches so that we can continue to count in integers;
2. by extending the concept of number to fractions and decimals and continuing to use a single unit of measure such as the yard.

No doubt the first method seemed the more natural—it was certainly the first to be tried—but as calculation became more prominent the limitations of this method became more obvious. In 1897 the weight of an Admiral's barge was given in a manual on seamanship as 1 ton 7 cwt 2 qtr 26 lb. Compare this with the following particulars of a motor car taken from a recent copy of a motoring journal: price £1207 1s. 1d., capacity 1485 c.c., weight 21¼ cwt, maximum speed 82·4 m.p.h. There has evidently been a swing towards the second method. One advantage of using a single unit is that the degree of accuracy is more obvious since it is merely necessary to note the number of significant figures. In ordinary everyday usage the same preference for a few units can be seen. A man's weight and height are given as 12 st 7 lb and 5 ft 10½ in. rather than as 1 cwt 4 st 7 lb and 1 yd 2 ft 10½ in., 5½ yd of cloth is more usual than 5 yd 1 ft 6 in., and a price is just as likely to be quoted as 37s. 6d. as in the form £1 17s. 6d. It is fair to say that teachers and examiners have been slow to follow this development and calculations like finding the cost of 43 things at £7 13s. 9½d. each have tended to persist in schools longer than anywhere else. No doubt such exercises provide excellent tests of speed and accuracy, but they delay the advance to matters of greater mathematical interest. The useful purposes served by these applications of arithmetic in the school course are to develop skills that may be needed when schooldays are over, and to deepen present interest by relating the work of the classroom to the real world outside. It is therefore particularly important here that exercises should be realistic and as far as possible based on material which is meaningful to children. The number of units used in a calculation should not exceed two, except in the case of money, and even there we need not always use three.

Applications to money, weights and measures begin with experience of the units, and the relationships between them. Money is particularly convenient as an introduction since experience can be gained by actually handling coins, although by this very fact it is limited to special fractions of a sum of money. On the other hand measures can be subdivided at will and this is well illustrated on the ruler where the child can find alternative sub-divisions of the inch. Thus linear measurement should be introduced early. Weights can also be handled but the idea of weight is not such an easy one. It is easy to underestimate the value of such experience in preparation for the learning of tables and conversion from one unit to another. Some teachers hold that this later work is simply the systematisation of previous experiences. No doubt in the case of the basic measures this experience will have been gained before the secondary school stage, but we may be well advised to remember its importance when we are undertaking remedial work or introducing new units. For example, it is very well worth while for the pupils to draw inch squares and actually handle inch cubes. If we seldom work with more than two units, conversion from one unit to another can be informal and the special setting-out of the old 'reduction' sums is not needed at all. Such parts of the tables as still need learning can be dealt with piecemeal as the need arises, and some of the traditional units can be omitted or mentioned only for their historical interest. In length measure, furlongs and chains are worth some attention, but the 'rod, pole or perch' is now obsolete. In square measure, the acre is the only special unit needed: its connection with furlongs and chains is worth noting. In weight measure, tons and hundredweights must be known, but quarters are not worth more than a mention. It should perhaps be more widely known that in America a hundredweight is 100 lb and a ton 2000 lb (but the 'metric ton' is 1000 kilogrammes and thus nearer to the English ton).

If pupils are to become thoroughly conversant with the metric system, it is highly desirable that they should start using it early. In this connection, scale drawing and graphical work using both the metric measures and inches with tenths, estimating a second decimal place when needed, can be invaluable. It is helpful, too, to use centimetre graph rulings as well as inch rulings.

The prefixes hecto-, deca-, deci-, and mega- should be introduced for completeness, and because of their occasional use, but regular practice should be confined to the commoner units. (See Appendix.)

A class is always interested to learn that the metric system originated in France after the French Revolution in a scientific attempt at reform, the French system before that having been as chaotic as our own. It never fails to amuse the pupils to hear that the metre, which was intended to be one forty-millionth of the circumference of the Earth, had to be otherwise defined in the end owing to an error in the original measurement. (See Appendix.)

The surveyor, the seaman, the silversmith, the chemist, the corn chandler and many others have their own units for their own purposes and some of these are worth mentioning as curiosities. Some too have interesting histories: thus the pennyweight was the weight of a silver penny, so naturally there were 240 pennyweights in a pound. But the days have gone by when schoolchildren were expected to know and use a wide variety of special tables.

Compound multiplication. Addition and subtraction of compound quantities will give little trouble beyond the correction of occasional lapses due to confusing the decimal scale with other scales. Compound multiplication may need more attention, but not necessarily all at once nor at the beginning of the course. We recommend that no single 'method' should be unduly emphasised but the habit of considering alternatives should be formed right from the start. Three general ways of organising compound multiplication can be suggested:

1. *Conversion to a single unit*. If the compound quantity is expressed in terms of the smallest unit, the calculation is reduced to the familiar one of multiplying integers. This is a thoroughly sensible procedure and must never be described as wrong. No doubt we shall take every suitable opportunity of pointing out the advantage of converting to a larger unit and using fractions. Some drill in the fractional parts of the shilling and pound may be advisable, otherwise the use of fractions may not be readily accepted. Later on, when logarithms are available, the advantage of approximate decimalisation becomes clear, but in the case of multiplication the use of this method is limited by the gross inaccuracy of the answer due to the multiplication of the error.

2. *Practice*. It has been customary to introduce this method at a compara-

tively late stage with a formal setting out that makes it one of the traditional 'sums' of school arithmetic. The method is commonly applied to questions involving money but even if decimal coinage is introduced we still recommend that the idea should be introduced informally at an early stage. For instance, a child who is accustomed to using kitchen scales knows that to weigh 2 lb 13 oz he needs the following weights: 2 lb, 8 oz, 4 oz, 1 oz, and so he might well set down the working of 2 lb 13 oz × 23 as follows:

	lb	oz
2 lb × 23	46	0
8 oz × 23	11	8
4 oz × 23	5	12
1 oz × 23	1	7
	64	11

A child who is still thinking in terms of coins might work a multiplication like £1 15s. 9d. × 23 as follows:

	£	s	d
23 pound notes	23	0	0
23 ten shilling notes	11	10	0
23 halfcrowns	2	17	6
23 ,,	2	17	6
23 sixpences		11	6
23 threepences		5	9
	41	2	3

No doubt 23 threepences would be expressed in shillings and pence by reckoning 4 threepences to the shilling first, but the simple connection with the previous line might be pointed out as a check, and can be adopted as a shorter method of calculation later on.

Then the widespread use of ready reckoners can be pressed into service. The class can be set to work in groups to make a ready reckoner for a milk roundsman when the price is, say, 8½d. a pint. The costs of 1 pint, 2 pints, 3 pints, ..., 9 pints; 10 pints; 20 pints, 30 pints, ..., are worked out and brought together. It is vital that these results should be correct and methods of checking them can be discussed. Thus the differences between the costs of 1, 2, 3, ... pints should be constant, and so should the differences between the costs of 10, 20, 30, ... pints. Also, and more usefully from our point of view, the cost of 20 pints should be 10 times the cost of 2 pints, and so on. Thus prominence is given to the relationship between different calculations. Or we might start with a price of 6d. per pint, work out the ready reckoner and then say, 'The price rises to 7d., have we to start our calculations all over again?' Thus we introduce the idea of calculating, say, 7d. × 23 as 6d. × 23 + 1d. × 23 Even with the simplest calculations it is worth showing the use of this idea as one possibility. For example, the cost of 27 books at £1 10s. each can be found as follows:

	£	s	d
Cost at £1	27	0	0
Cost at 10s.	13	10	0
Cost at £1 10s.	40	10	0

Often the idea comes as a pleasant surprise and is taken up with enthusiasm, especially as there is scope for ingenuity in choosing aliquot parts. On the other hand, less able pupils sometimes find this choice a difficulty and it may be advisable to discuss the method of calculation in a large number of cases which are not fully worked out.

3. *Long multiplication*. It is doubtful whether there is much point in teaching any particular formalised setting out, and there is no case at all for teaching one as *the* method of multiplication. This attributes more importance to the position of figures on a page than to principles and spreads the view that mathematics is a matter of knowing the right rules rather than of applying commonsense.

Compound division. This can be a complicated process and is now little used, but two points may be noted.

1. It is often convenient to reduce the dividend to a single unit, but this can never be a rigid rule.
2. If the divisor is a small number, or easily factorised, short division will often be preferable to long, whereas if it is a larger or more awkward number, then long division or logarithms and a single unit should be used.

For instance, if a farmer floods his field for skating with 931 000 gallons of water and is charged £10 6s. 6d. by the Water Board, the cost per 1000 gallons is

$$\frac{2478}{931}\text{ pence}\simeq 2{\cdot}66\text{ pence};$$

but if 4 boys share the cost of a camp, which amounted to £16 12s. 8d., then each will pay

$$\frac{\pounds 16\text{ 12s. 8d.}}{4}=\pounds 4\text{ 3s. 2d.}$$

Pupils should be encouraged, from their early days, to discuss the possible methods and exercise their own choice in deciding which to use.

Decimalisation of quantities in compound units. Some practice in decimalisation is necessary both in readiness for division and also for the use of logarithms. There are special methods for rapidly expressing money as a decimal of £1 to a limited number of places, but it is better to rely on the direct method which is of general application.

Express £2 3s. 5d. *as a decimal of* £1.

$$\begin{array}{r l} 12\,\underline{)\,5{\cdot}00} & \text{d.}\\ 20\,\underline{)\,3{\cdot}41666} & \text{s.}\\ 2{\cdot}17083 & \pounds \end{array}$$

∴ £2 3s. 5d. = £2·1708, to 4 decimal places.

Express 3 *tons* 550 *lb in tons to* 4 *sig. fig.*

$$3\tfrac{55}{224}=3{\cdot}2455\ \dots\text{ by long division.}$$

∴ 3 tons 550 lb = 3·246 tons to 4 sig. fig.

The converse process is also needed.

Write £2·171 *in £ s. d. to the nearest penny.*

$$\begin{array}{rl} \pounds 2{\cdot}171 & \\ 20 & \\ \hline 3{\cdot}42 & \text{s.} \\ 12 & \\ \hline 5{\cdot}04 & \text{d.} \end{array}$$

∴ £2·171 = £2 3s. 5d. to the nearest penny.

The above examples show the methods of calculation advocated but it is not suggested that any particular way of setting down the working is necessarily the best.

INSTITUTE OF EDUCATION FOR WOMEN
Dept. of Extension
SERVICE.
CALCUTTA-27

CHAPTER 6

FRACTIONS AND DECIMALS

FRACTIONS have grown out of a need felt increasingly through the ages to measure quantities more accurately. At first this need was met by introducing smaller units in order to avoid the use of fractions. This is understandable when it is remembered that even computation with integers used to be regarded as a special skill. Nevertheless, fractions of one sort and another have been used at least since the time of the Ahmes Papyrus (*c.* 1550 B.C.). There was a tendency among the ancients to use only unit fractions, i.e. those with a unit numerator. This led to expressions (in modern notation) such as

$$\frac{1}{42}+\frac{1}{86}+\frac{1}{129}+\frac{1}{301}$$

for the fraction $\frac{2}{43}$. Another system of fractions used from early times was the sexagesimal system which we still use for fractions of a degree, e.g. 21° 35′ 46″, but further places were used in the Middle Ages. To distinguish between fractions in the form $\frac{a}{b}$ and sexagesimal fractions, the former were called 'common fractions', *fractiones vulgares* in Latin, and hence 'vulgar fractions'.

Decimal fractions may be said to have been invented in the sixteenth century A.D. by Christoff Rudolff but his work does not appear to have been appreciated. Fifty-five years later, in 1585, Stevin published an account of how decimal fractions could be used for all business purposes.

Common and decimal fractions serve the same purpose and are interchangeable in that every common fraction may be expressed as a terminating or recurring decimal, and every terminating or recurring decimal can be expressed as a common fraction. From this it might appear that one system is redundant but each has its uses: *common fractions* represent rational numbers simply, they are well adapted to exact calculation, and they link well with the idea of ratio; *decimal fractions* enable comparisons between fractions to be made at sight and so are appropriate for recording measurements, and they have the advantage that the four rules are basically the same as for integers.

It cannot be laid down whether common or decimal fractions should be introduced first. At the time of writing, infants have some experience of simple common fractions and they are more likely to wish to record such fractions rather than those involving tenths. Their use of any mathematical notation should grow out of a wish to express their experience in written form and, with the introduction of decimal coinage, there will be a tendency for the decimal notation to be used earlier than is general at present. Neither notation should be taught to the exclusion of the other but rather the study of one type of fraction should enrich the understanding of the other.

Fractions

The comparison of simple fractions is a useful exercise. In this and in addition of fractions, stress should be laid on the principle involved in bringing fractions to a common denominator, viz. that a fraction is unaltered if its numerator and denominator are multiplied by the same number. At first fractions should be written with the same denominator separately before the fractions are combined. In algebra, too, pupils starting to add fractions will find the process easier to understand if they are accustomed to writing the denominators separately. In general, mixed numbers should not be reduced to improper fractions in addition and subtraction.

Computation with fractions should be simple: it is not easy to find the need in mathematics even for the addition of, say, $\frac{2}{3}$ and $\frac{5}{8}$. It is harder still to find the need for the addition or subtraction of three or four fractions until the calculus is reached and definite integrals of polynomials have to be evaluated.

The rules for manipulating fractions are not merely a logical extension of those for integers. The teacher should be aware of this as it may affect his attitude and method of approach. From the logical point of view, a fraction is a pair of numbers and rules for the manipulation of such pairs could be laid down in a variety of ways. One particular way is chosen because it fits in with practical requirements and this is a justification for the usual classroom approach in which the rules are built up from practical examples.

A first consideration of the four rules for fractions should grow out of practical and, if possible, concrete situations. The primitive ideas of addition and subtraction as 'putting together' and 'taking away' are easily carried over into situations where quantities are expressed as fractions. But this is not the case with multiplication and division by fractions. The child who is only used to multiplication by integers thinks of the process as one which 'makes bigger' and multiplication by a fraction may appear as a contradiction in terms. This means that, although there is logically a need to define addition of and multiplication by fractions, it is much easier to give a psychologically satisfying explanation of the former process.

In dealing with integers, we proceed from a statement such as '2 oranges + 3 oranges = 5 oranges' to the abstract '$2+3=5$', and a similar method is necessary for fractions. To show the meaning of a fraction $\frac{3}{4}$, we may divide a length, area or sum of money into four equal parts and take three of them. It is easy to see that if we divide into eight equal parts and take six of them the result is the same. From this we deduce the abstract statement that $\frac{3}{4}=\frac{6}{8}$. From the fact that, if three of the eight parts are added to two more, the result is five such parts we conclude that $\frac{3}{8}+\frac{2}{8}=\frac{5}{8}$. In such a way we build up the rule for addition of fractions.

The need for multiplication by a fraction may be suggested in a number of ways. In each of these there will arise a stage where multiplication by a fraction is defined. Two suggestions are given below.

Areas. If a class is familiar with areas of rectangles whose lengths and breadths are integral numbers of units, they may be asked to find the areas of rectangles $\frac{1}{3}$ in. by 2 in., 2 in. by $\frac{1}{3}$ in., $\frac{1}{2}$ in. by $\frac{1}{3}$ in.

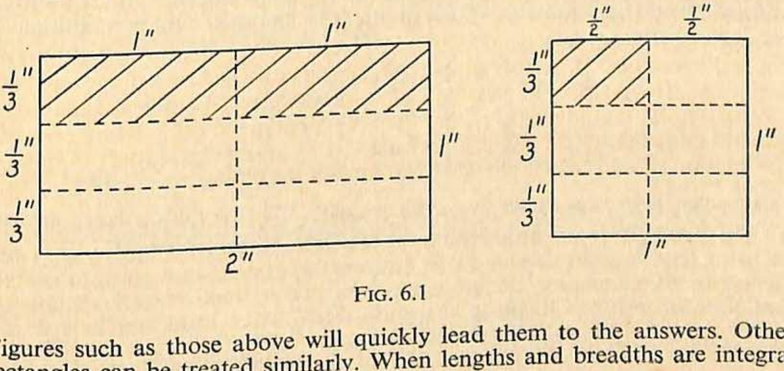


FIG. 6.1

Figures such as those above will quickly lead them to the answers. Other rectangles can be treated similarly. When lengths and breadths are integral numbers of units, the area is found by *multiplication* and so the class may be led to enlarge their concepts of multiplication to include multiplication by fractions. An advantage of this method is that it can be illustrated visually in the classroom.

Commercial examples. This approach has been used widely as a means of introducing the rules for multiplying and dividing by fractions. The approach depends on proportion. The class consider some such illustration as the buying of cloth. It is evident that the cost of 5 yards is 5 times the cost of 1 yard, i.e. the cost of one yard multiplied by 5. For the cost of $\frac{3}{4}$ yard we divide the cost of one yard by 4 and multiply by 3, i.e. we take $\frac{3}{4}$ of the cost of one yard. It is convenient to call this 'multiplication' as in the other case. Thus multiplication by $\frac{3}{4}$ is identified with taking '$\frac{3}{4}$ of' the cost of one yard.

The fraction $\frac{15}{5}$ (fifteen fifths) may be simplified to 3 (three wholes) and a comparison with '$15 \div 5 = 3$' leads, with similar examples, to a realisation that the fraction bar may be interpreted in two ways: as part of the notation that we use for writing fractions, and as an instruction to divide. In general it is better to use the fractional notation rather than the division sign. Logically, the rational number $\frac{15}{5}$ or $\frac{3}{1}$ is distinct from the integer 3 because the rational numbers are *pairs* of integers. In practice, however, the distinction is not important since a fraction such as $\frac{3}{4}$ is represented by dividing something into 4 equal parts and taking 3 of them.

Pupils entering the grammar school are usually familiar with the rule for division, 'To divide by a fraction, turn it upside down and multiply,' but they can seldom explain why it works. Perhaps the most direct method is to say, for example, that

$$\frac{\frac{3}{5}}{\frac{2}{7}} = \frac{\frac{3}{5} \times \frac{7}{2}}{\frac{2}{7} \times \frac{7}{2}} = \frac{3}{5} \times \frac{7}{2}.$$

Alternatively, both fractions may be expressed with the same denominator, e.g.

$$\frac{\frac{3}{5}}{\frac{2}{7}} = \frac{\frac{21}{35}}{\frac{10}{35}} = \frac{21}{10}.$$

Another explanation, which is favoured by many teachers, is to multiply numerator and denominator of the quotient by the least common multiple of the denominators, e.g.

$$\frac{\frac{3}{5}}{\frac{2}{7}} = \frac{\frac{3}{5} \times 7 \times 5}{\frac{2}{7} \times 7 \times 5} = \frac{3 \times 7}{2 \times 5} = \frac{21}{10}.$$

This looks longer on paper but with practice the two middle steps are not written down and it is easy to simplify fractions by this process, e.g.:

$$\frac{2\frac{1}{2}}{3\frac{1}{2}} = \frac{5}{7} \quad \text{and} \quad \frac{\frac{1}{a} + \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}} = \frac{b+a}{b-a}.$$

There is a difference of opinion among teachers as to the wisdom of allowing the numerator and denominator to be scratched through when 'cancelling' is done. It is recommended, however, that crossings out should be kept to the minimum.

$$\frac{2}{14}=\frac{1}{7}$$

is clearer than

$$\frac{\overset{1}{\cancel{2}}}{\underset{7}{\cancel{14}}}=\frac{1}{7}.$$

and good results can be obtained in more complicated examples without any crossing out, e.g.

$$\frac{7}{13}\times\frac{9}{14}\times\frac{26}{12}=\frac{1}{1}\times\frac{3}{2}\times\frac{2}{4},$$

$$=\frac{3}{4}.$$

For the first step of this method the lines of the fractions and the multiplication signs are written first and the figures then inserted in the appropriate spaces. When all the spaces are used up another step may be needed if there is further cancelling to be done, but this can often be disposed of mentally as in the above example.

If crossing out is permitted, plenty of space is required above and below, and the work must be kept neat and legible. Mistakes in accuracy are frequently caused by small or carelessly written figures during this operation. Checking is easier if different types of stroke are used for the different stages of the 'cancellation', e.g.

$$\frac{\overset{1}{\cancel{7}}}{\underset{1}{\cancel{13}}}\;\frac{\overset{3}{\cancel{9}}}{\underset{2}{\cancel{14}}}\;\frac{\overset{2}{\cancel{26}}}{\underset{4}{\cancel{12}}}=\frac{3}{4}.$$

In no event should pupils be allowed to cross out a number by drawing a separate line through each digit. Difficulties of printing lead to numbers being crossed out in this way in the examples shown in some books.

In connection with fractions the pupil is brought up against the 'associative' and 'distributive' laws of algebra. Thus

$$3\times 2\tfrac{1}{5}=6\tfrac{3}{5},\quad\text{but}\quad 3\times 2\times\frac{1}{5}=6\times\frac{1}{5},\quad\text{not}\quad 6\times\frac{3}{5};$$

$$\text{and}\quad \frac{50\times 3}{60}=\frac{5\times 3}{6},\quad\text{but}\quad \frac{50+3}{60}\quad\text{is not equal to}\quad \frac{5\times 3}{6}.$$

'Associative' and 'distributive' are long and repellant words which need not be used in the classroom; if any names are needed for these laws they may be referred to as the 'any-order rule' and the 'do-it-to-each rule'. Wrong ideas in this kind of work persist to a much later stage: for example, when dealing with area of a triangle, pupils are sometimes found to be uncertain whether *half base × height* is the same as $\frac{\text{base}\times\text{height}}{2}$. The habit of 'cancelling' sometimes degenerates into the crossing out of any two numbers that are the same, e.g.

$$\frac{a+x}{a}=x, \quad \text{or} \quad (1+x).$$

Such errors occur in the work of quite advanced pupils and teachers must always be prepared to refer back to such simple illustrations as

$$\frac{20\times 30}{10}=2\times 30 \quad \text{and} \quad \frac{20+30}{10}=2+3.$$

Geometrical illustrations may also be used. Thus we can double the perimeter of a rectangle by doubling both the length and the breadth; but to double the area we double only one of them.

Decimals

Decimal fractions are being used more and more widely in the world. They lend themselves readily to computation with tables, slide rules, calculating machines and computers, and this is probably the main reason for their advancement as against common fractions. For similar reasons, decimal coinage and the metric system of measurement are also used far more extensively than at the time of the 1932 Report. Children are likely to have acquired some knowledge of the decimal system of notation at quite an early age, and it is on the basis of such knowledge that a formal treatment of the use and manipulation of decimals can be built.

There is no need to link the introduction of decimals with the metric system. For instance, an exercise in geometry might lead to the measurement of a length in inches and the convenience of the notation can be pointed out. If the class are able to estimate a length to the nearest hundredth of an inch, the notation can be extended to two decimal places: there is no point in working with more than two decimal places in the early stages. A natural use for the second decimal place occurs in finding areas by counting squares on a $\frac{1}{10}$ in. grid.

Place value is the aspect of the decimal notation which makes it so easily used in machine methods of computation but, although place value is so important, it tends to be forgotten as new processes and ideas are introduced, and it is essential that attention be drawn to it from time to time. Some teachers find that calculating machines provide an effective way of doing this; others introduce the binary and other scales for the same reason. At some stage numbers should be written in column form (hundreds, tens, units) and operated on, e.g. multiplied by 10 and the figures moved into the appropriate columns. Then multiplication by 100 can be performed and the consequent movement of figures noted. Extension of the columns to the left (i.e. thousands, ten-thousands, etc.) will be necessary and should help with understanding. This can be followed by examples of division by 10 and 100, with the resulting movement of figures towards the right. Finally, after enough examples of multiplication and division by powers of 10, the pupil should be able to form the rules for himself and the extension of the columns to the right (tenths, hundredths) presents little difficulty.

Addition and subtraction are easy; the only idea to be stressed is that, with the aid of the decimal point, the place value of the digits must be meticulously preserved.

$$\begin{array}{rr}
\textit{Addition} & \textit{Subtraction} \\
13{\cdot}5 & 82{\cdot}5 \\
2{\cdot}48 & 4{\cdot}79 \\ \cline{2-2}
0{\cdot}07 & 77{\cdot}71 \\ \cline{2-2}
1{\cdot}1 & \\ \cline{1-1}
17{\cdot}15 & \\ \cline{1-1}
\end{array}$$

The close connection between fractions and decimals can be demonstrated by the conversion of one into the other, e.g.

$$\frac{1}{4}=1\div4,\qquad 4\,\overline{)\,1{\cdot}00}\atop\quad 0{\cdot}25$$

and

$$0{\cdot}075=\frac{75}{1000}=\frac{3}{40}.$$

But many fractions are not convertible into exact decimal forms and this should lead to profitable discussion on the number of decimal places required or usefully appropriate. In this connection (or, for example, with ratio problems) conversion of fractions such as $\frac{2}{3}$, $\frac{7}{11}$, $\frac{3}{7}$ to decimal notation will be necessary, and will lead the pupils to discover recurring decimals. These provide a first encounter with infinite series which have finite sums, and are usually regarded as great fun for a time. Decimal equivalents of the simpler fractions (e.g.: $\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{1}{3}$, $\frac{2}{3}$) should be memorised and, until decimal coinage is introduced, such simple equivalents as 1s. = £0·05, but on the whole it is advisable to use the straightforward method of conversion shown on page 18.

Multiplication. In the past, several different methods for the multiplication of decimals have been suggested and tried out. These were discussed at some length in the 1932 Report, as at that time opinion was much divided as to which was the best method. Opinion now seems to have crystallised in favour of that which treats the calculation of the figures and the establishment of their place value separately. This method is arrived at directly from the meaning of a decimal. As soon as it is realised that $2{\cdot}73=\frac{273}{100}$, and $15{\cdot}6=\frac{156}{10}$ it follows that

$$2{\cdot}73\times15{\cdot}6=\frac{273}{100}\times\frac{156}{10}=\frac{273\times156}{1000}=\frac{42\,588}{1000}=42{\cdot}588.$$

The numbers multiplied have 2 decimal places and 1 decimal place, the product 2 + 1 = 3 places, and so the rule is arrived at, 'Multiply the digits as if they were whole numbers and count the total number of decimal places.'

Practice with simple cases is specially important, e.g. to find $(0{\cdot}02)^3$. This has 6 decimal places of which the last is 8. The answer is 0·000 008. Similarly, $0{\cdot}04\times0{\cdot}003=0{\cdot}000\,12$ and $0{\cdot}037\times0{\cdot}03=0{\cdot}001\,11$.

Longer examples can be set down thus

$$\begin{array}{r}753\\182\\\hline 753\\6024\\1506\\\hline 137046\\\hline\end{array}$$

$$7{\cdot}53\times18{\cdot}2=137{\cdot}046.$$

$$\left\{\begin{array}{c}\text{Total number of decimal places}\\2+1=3\end{array}\right\}^{*}$$

The decimal point is inserted in the answer in accordance with [illegible] bearing in mind that a zero which occurs at the end of the product does [illegible] as one of the digits after the point even though it may subsequently be discarded, e.g.

---

* Not written down.

Notes. (1) The decimal point is written as a full stop in the U.S.A. and as a comma on the Continent; in both the multiplication dot is written as we write our decimal point. This variation in practice is illustrated below:

	British	*U.S.A.*	*Continental*
Decimal point	4·5	4.5	4,5
Multiplication dot	7 . 8	7·8	7·8

(2) Method of rough estimate:

$$23{\cdot}61 \times 42{\cdot}1 \simeq 20 \times 40 = 800.$$

$$\frac{0{\cdot}00724}{0{\cdot}0892} = \frac{0{\cdot}724}{8{\cdot}92} \simeq \frac{0{\cdot}7}{9} \simeq 0{\cdot}08.$$

When the pupil can rely on a rough approximation to fix the position of the decimal point, the point may be ignored in the working and inserted in the answer. Those who use the slide rule do this.

(3) A pupil who can recognise 0·125 as $\frac{1}{8}$ will find it easier to multiply by $7\frac{1}{8}$ rather than by 7·125.

(4) Pupils should be given a good deal of practice with multiplication and division by such numbers as 30 or 700. Simple calculations with easy numbers should be used frequently to establish fluency with basic ideas.

(5) The kind of question requiring a remainder, e.g., 'How many pieces of wire each 1·6 in. can be cut from a yard, and how much is left over?' is probably best done by changing each length into tenths of an inch; the problem then becomes 'How many pieces of wire each 16 tenths of an inch can be cut from 360 tenths, and how much is left over?' and is then a simple division.

Historical note.* The decimal point has an interesting history. It was not the earliest notation for representing decimals and its advantages were not immediately recognised. Indeed for a hundred years or more it was just one of a number of notations in common use. In 1543 Robert Recorde wrote the result of dividing 1253 by 10 as $125\underline{|3}$, i.e. quotient 125 and remainder 3. This was not used by Recorde as a notation for decimals but was the source of a later notation.

In 1585 the Dutchman Simon Stevin published a book to popularise decimals and he used two notations. The number 123·456 he wrote as (i) $123'4''5'''6''''$ or (ii) 123⓪4①5②6③. The decimal figures 456 were called prime, second, third, and it is interesting to compare these names with the Latin names for minutes and seconds of angle: pars minuta *prima* and pars minuta *secunda*. The modern signs for minutes and seconds may be compared with Stevin's first notation. Stevin's second notation is clearly a forerunner of index notation, and the zero indicating the unit place is particularly interesting. He is in fact writing 1, 10, 100, 1000 as 10^0, 10^1, 10^2, 10^3. Both notations are cumbersome and show no appreciation of the fact that it is only necessary to indicate the place value of one digit. Stevin may have been influenced by his desire to expound the basic principles in which place value is all important. For example in an addition he uses his symbols at the head of the columns:†

$$\begin{array}{cccccc}
 & & \textcircled{0} & \textcircled{1} & \textcircled{2} & \textcircled{3} \\
 & 2 & 7 & 8 & 4 & 7 \\
 & 3 & 7 & 6 & 7 & 5 \\
8 & 7 & 5 & 7 & 8 & 2 \\
\hline
9 & 4 & 1 & 3 & 0 & 4
\end{array}$$

---

* Based chiefly on F. A. Yeldham, *The teaching of arithmetic through* 400 *years*, (Harrap).

† From D. E. Smith, *History of mathematics*, (Ginn & Co.), Vol. II, p. 243.

$$\begin{array}{ll} 2{\cdot}48\times 7{\cdot}5 & \begin{array}{r} 248 \\ 75 \\ \hline \end{array} \\ =18{\cdot}600, & \begin{array}{r} 1240 \\ 1736 \\ \hline \end{array} \\ =18{\cdot}6. & \begin{array}{r} 18600 \\ \hline \end{array} \end{array}$$

Division. For division, the method of setting down the problem in 'fraction bar' form and then converting the divisor to a whole number is recommended, e.g.:

$$7{\cdot}52\div 3{\cdot}7=\frac{7{\cdot}52}{3{\cdot}7}=\frac{7{\cdot}52\times 10}{3{\cdot}7\ \times 10}=\frac{75{\cdot}2}{37}$$

The second step can be left out as soon as the process is understood and the solution can be completed either by cancelling when appropriate (and a number of such examples should be taken until the idea is grasped) or by the usual methods of short or long division, e.g.:

$$\frac{3{\cdot}36}{1{\cdot}2}=\frac{3{\cdot}36\times 10}{1{\cdot}2\ \times 10}=\frac{33{\cdot}6}{12}=2{\cdot}8.$$

$$\frac{0{\cdot}0192}{0{\cdot}008}=\frac{0{\cdot}0192\times 1000}{0{\cdot}008\ \times 1000}=\frac{19{\cdot}2}{8}=2{\cdot}4$$

$$\frac{67{\cdot}2}{0{\cdot}32}=\frac{67{\cdot}2\ \times 100}{0{\cdot}32\times 100}=\frac{6720}{32}=\frac{840}{4}=210.$$

$$4{\cdot}0068\div 371=0{\cdot}0108. \qquad \begin{array}{r} 0{\cdot}0108 \\ 371\,\overline{)\,4{\cdot}0068} \\ 3\,71 \\ \hline 2968 \\ 2968 \\ \hline \end{array}$$

Find $0{\cdot}007\,24\div 0{\cdot}0892$ *to three places.*

$$\frac{0{\cdot}007\,24}{0{\cdot}0892}=\frac{72{\cdot}4}{892}\simeq 0{\cdot}081 \qquad \begin{array}{r} 0{\cdot}0811 \\ 892\,\overline{)\,72{\cdot}4000} \\ 71\,36 \\ \hline 1\,040 \\ 892 \\ \hline 1480 \end{array}$$

Decimal places, significant figures, accuracy and approximations are considered in Chapter 13.

Standard form. A number given as $a\,.\,10^m$ is said to be in 'standard form' when a lies between 1 and 10 and m is an integer. This form is specially useful to express numbers which are either very large or very small, such as are frequently met in science; thus we may say,

'The Sun's distance from the Earth$\simeq 9{\cdot}3\times 10^7$ miles,'

'The mass of an electron$\simeq 9\times 10^{-27}$ grammes.'

At some stage the connection with the characteristics of logarithms should be brought out.

In 1631 William Oughtred went back to Recorde's symbol for division by 10 and wrote 123·456 as $123\,\underline{|\,456}$. Other writers about this time used the vertical line only, calling it the *prime line* at first and later on the *virgula* (little rod). Many writers felt the need to indicate the decimal figures and some did it by putting a dot over each one. In 1616 Kepler used 123,456 and 123(456), and Napier at about the same time used the comma and a variant of Stevin's notation, 123,4′5″6‴. In 1677 Cocker's Arithmetic mentioned that the following notations were in use and recommended the first as 'most fit for calculation': 123·456, 123 | 456, $123\,\underline{|\,456}$, $12\dot{3}456$, $12\overset{|}{\underset{\backslash}{3}}456$, 123,456, $123,\dot{4}\ddot{5}\dddot{6}$. In the subsequent years the decimal point gradually ousted the other notations in England, but it is interesting to know that on the Continent the comma (Cocker's sixth notation) is used instead of the point (see Note 1). There is something to be said for Cocker's fourth and fifth notation. The marking of the unit digit by a dot placed after it obscures the symmetry with respect to the unit place of multiplication and division by the same power of 10, e.g.

$$2\times 10=20, \qquad 2\div 10=\cdot 2,$$
$$2\times 10^2=200, \qquad 2\div 10^2=\cdot 02.$$

With Cocker's fourth form the symmetry is obvious:

$$2\times 10=2\dot{0}, \qquad 2\div 10=\dot{0}2,$$
$$2\times 10^2=20\dot{0}, \qquad 2\div 10^2=\dot{0}02.$$

Many mathematicians including Augustus De Morgan (1850) have pointed out that such a notation would greatly simplify the rules for determining the characteristic of a logarithm. De Morgan advocated the rule 'The characteristic of (the logarithm of) a number is the number of places by which the first significant figure is distant from the unit's place; and is positive when that first figure falls to the left, negative when to the right'.

CHAPTER 7

PERCENTAGES

THE term *per cent* (probably derived from the Italian *per cento*) was first used in connection with interest. (Gresham in 1568 wrote of 'Th'interest of xij.per cent by the yeare'.) The modern use of percentages is much wider and the literal meaning of '12 for every hundred' has been slightly modified. The practical effect of taking 12 for every hundred is to produce an amount equal to $\frac{12}{100}$ of the original. Thus a percentage is simply a fraction with denominator 100. The advantage of expressing fractions with such a standardised denominator is that they can be easily compared one with another. The choice of 100 is of course arbitrary and for some purposes (e.g. population) 1000 is more commonly used. Decimals serve the same purpose, and percentages may be regarded as a half-way house between fractions and decimals.

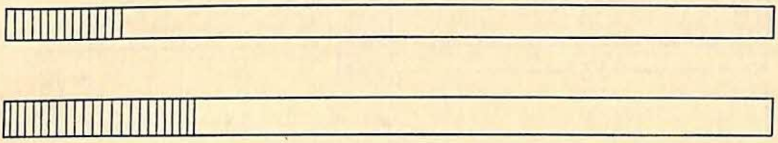


FIG. 7.1

In Fig. 7.1 the shaded parts of the two rectangles are respectively $\frac{3}{20}$ and $\frac{1}{4}$ of the areas. To compare them, it is customary to imagine the length of each rectangle to be divided into 100 equal parts and to say that the shaded portions are respectively 15 parts and 25 parts, or '15 per cent' and '25 per cent' of the whole. The decimals 0·15 and 0·25 express the same thing just as well, but the phrases '15 per cent' and '25 per cent' are commonly used and convey the meaning more clearly to many people.

At an early stage of the school work on percentages pupils should become familiar with the percentage and decimal equivalents of the simpler fractions. Diagrams may be used with advantage (Fig. 7.2).

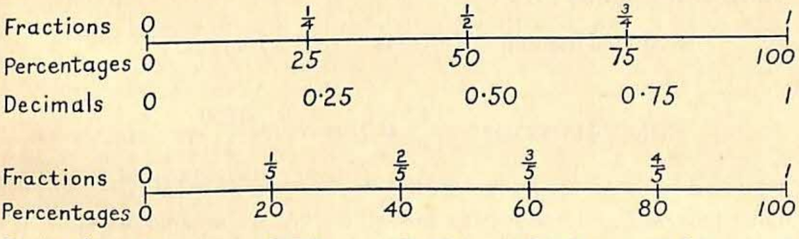

Fractions 0 1/4 1/2 3/4 1
Percentages 0 25 50 75 100
Decimals 0 0·25 0·50 0·75 1
Fractions 0 1/5 2/5 3/5 4/5 1
Percentages 0 20 40 60 80 100
Decimals 0 0·20 0·40 0·60 0·80 1

FIG. 7.2

Interchange between the fractional, decimal and percentage forms must be practised, both in simple cases taken orally and in those for which some written work is necessary. This work, with simple applications, should be allowed to become familiar before any mention is made of percentage increases and decreases.

Besides questions of the type

Find 3 *per cent of* £12

and the corresponding inverse questions, the applications should include some in which the actual amounts are not known, such as

> *If* 63·8 *per cent of the world's oil comes from North and Central America, what percentage comes from the rest of the world?*

and some on overlapping classes, such as

> *If* 56 *per cent of a school have had measles and* 38 *per cent have had chicken-pox and* 12 *per cent have had neither, what percentage have had both?*

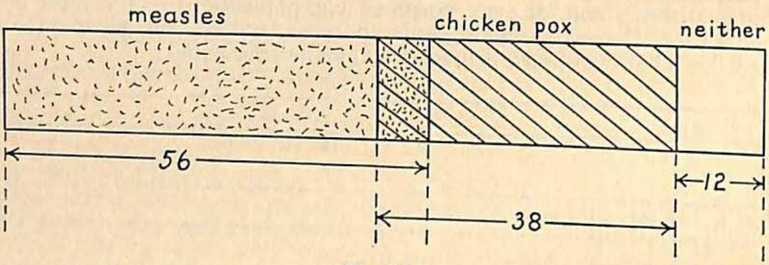


FIG. 7.3

Questions of the type

If 12 *electric lamps out of* 55 *are defective, what percentage is this?*

present some difficulty in expression. The pupil must ask himself, 'What fraction of the lamps are defective?' and should write the answer as

$\frac{12}{55}$ of the lamps are defective, i.e. 0·218 … of the lamps.

∴ 21·8 per cent are defective.

Alternatively he may write

$$\text{Required fraction} = \frac{12}{55} = 0{\cdot}218\ \ldots = 21{\cdot}8 \text{ per cent.}$$

or

$$\text{Required percentage} = \frac{12}{55} \times 100 \text{ per cent} = \frac{1200}{55} \text{ per cent,}$$
$$= 21{\cdot}8 \text{ per cent.}$$

Questions on profit and loss should be treated on the same principle. If a man buys a house for £2000 and sells it for £2400, his percentage profit is $\frac{400}{2000} \times 100$ per cent, i.e. 20 per cent. But it should be mentioned that the shopkeeper does not reckon his percentage profit in this way. For him the retail price is commonly fixed first and is thus the 'original value'. The wholesaler allows him a discount of so much per cent on this value. The discount is his profit, and it thus becomes natural for him always to compare his profit with his selling price.

A bookseller buys a 15*s. book from the publisher at a discount of* 30 *per cent. What profit does he make on each copy?*

Profit = discount = $\frac{30}{100} \times$ 15s. = 4s. 6d.

When increases and decreases are considered (this will be at a later stage, after some knowledge of ratio has been gained), practice should first be given in the direct calculation of the increase or decrease or of the new value. The simplest cases of all, when the original value is 100 units or 1 unit, should be used to introduce the idea that, for example, an increase of 8 per cent is equivalent to multiplication by $\frac{108}{100}$ or 1·08. Results should also be expressed in the ratio form as

Original value : increase : new value
= 100 : 8 : 108,

and by means of diagrams (Fig. 7.4).

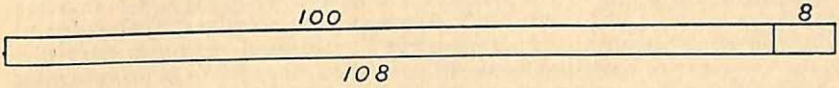


FIG. 7.4

Similarly, for a decrease of 8 per cent,

Original value : decrease : new value
= 100 : 8 : 92.

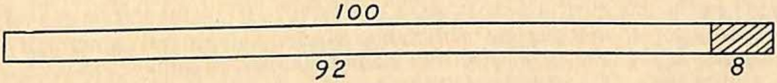


FIG. 7.5

Many oral questions should be asked of the type

Express the original value (or the increase) as a fraction of the new value.

After this, inverse questions should present less difficulty. For example

In June 1961, 230 000 *overseas visitors came to Britain, an increase of* 11 *per cent on the number for June* 1960. *How many came in June* 1960*?*

We must first ask the question, 'Per cent of what?' i.e. we must make it clear what the sentence means. The general principle is that the increase is compared with the original value, i.e. the value just before the increase; in this case, with the number for June 1960.

Original value : increase : new value
(June 1960) (June 1961)
= 100 : 11 : 111.

The number for June 1960 was therefore $\frac{100}{111} \times$ 230 000, i.e. about 207 000.

In a certain year, 5600 *copies of a book were sold, a decrease of* 20 *per cent on the previous year. How many were sold in the previous year?*

The decrease is 20 per cent of the number sold in the previous year.

$$\begin{array}{lccccc} & \text{Original value} & : & \text{decrease} & : & \text{new value} \\ & \text{(previous year)} & & & & (5600) \\ = & 100 & : & 20 & : & 80. \end{array}$$

Therefore the number sold in the previous year was $\frac{100}{80} \times 5600$, i.e. 7000.

Index numbers are an important example of percentages. For example, the index of industrial production for the United Kingdom, taking the year 1954 as 100, was 106 in 1958, 113 in 1959, and 120 in 1960. This means that in 1958 production was 6 per cent above the 1954 level, in 1959 13 per cent above, and in 1960 20 per cent above. By this means we are able to compare the production for one year with that of another more clearly than if we were given the actual values.

CHAPTER 8

RATIO AND RATE

Quantities of the same kind can be compared in two ways: by difference and by ratio. Near the end of a race a competitor is concerned with the distances that separate him from the other runners, i.e. with differences; but it would serve little purpose to say that the model of a ship was 500 feet shorter than the ship itself: it is more useful to compare them by scale, i.e. by ratio. Both these methods of comparison are in everyday use and are to that extent familiar to children. They are accustomed on the one hand to such phrases as 'two inches taller', 'sixpence more pocket money'; and on the other to 'twice as much milk', 'half as long again', 'we're half way there'.

Ratio is used whenever we answer the question, 'How many times?' To say that a football ground is 1½ times as long as it is broad is to state a ratio; and to say that its breadth is 80 yards, i.e. 80 times the length of the standard yard, is again to state a ratio. Thus every measurement is a ratio, the given quantity being expressed as so many times a standard quantity.

A ratio compares quantities of the same kind (i.e. quantities capable of being measured in the same units), but we sometimes need to express a connection between quantities of different kinds as, for example, when we measure speed in miles per hour or state a price in shillings per hundred. These are examples of the idea of *rate*. But a rate does not necessarily connect quantities measured in different units, e.g. we speak of the standard rate of income tax being 7s. 9d. in the pound. Here the idea is that, for every pound of income in excess of a certain amount, 7s. 9d. tax is to be paid. Rates generally include the phrase 'in the', the word 'per' (from mediaeval Latin, meaning 'for every' or 'for each') or, more briefly, 'a', implying that the magnitude of one quantity will be determined from the other, using the given rate. Thus the price of eggs 3s. 9d. a dozen implies that 2 dozen will cost 7s. 6d., ½ dozen 1s. 10½d., and so on. In short, a ratio is used to connect two quantities, but a rate implies a functional relationship and is used to connect corresponding values of two variables.

Consider for example the following sets of numbers:

Distance in yd.	50	5050	7050	10 050
Clock time (a.m.)	11.00	11.05	11.07	11.10

No comparison between these two sets has any meaning until we know of some connection between them which enables us to pair them off in a (1, 1) correspondence. For example, if the upper set gives the distance of a vehicle from a fixed point at the times given by the lower set, then we can match the sets as shown in the following diagram:

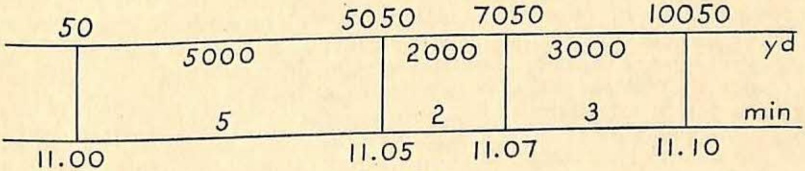


FIG. 8.1

Considering any two intervals in one line, and the corresponding intervals in the other line, we discover an equality of ratios:

$$\frac{5000\text{ yd}}{2000\text{ yd}}=\frac{5\text{ min}}{2\text{ min}};\quad \frac{2000\text{ yd}}{3000\text{ yd}}=\frac{2\text{ min}}{3\text{ min}};\quad \frac{3000\text{ yd}}{5000\text{ yd}}=\frac{3\text{ min}}{5\text{ min}}$$

But we can also match corresponding intervals across the lines as follows:

$$\frac{5000}{5}=\frac{2000}{2}=\frac{3000}{3}=1000$$

We say that this indicates that distance is covered at the constant rate of 1000 yd per min.

An equality of ratios is described as a *proportion*. In the table

Income	£500	£600	£700
Tax	£100	£120	£140

we note that $\frac{£600}{£500}=\frac{£120}{£100}$ and that $\frac{£700}{£600}=\frac{£140}{£120}$. We describe this by saying that the tax is proportional to the income. It is also seen that $\frac{£100}{£500}=\frac{£120}{£600}=\frac{£140}{£700}$, i.e. the rate of tax is constant. Thus when two variable quantities are in proportion, there is a constant rate as well as equality of ratios.

In everyday language the word proportion is used with different meanings in such phrases as 'a proportion of the cost' and 'the proportions of the human body'. Time is well spent in examining such usages and contrasting the mathematical and more general uses of the word.

Examples of the type *Find the cost of* 3 *yards at* 1*s.* 9*d. a yard* are examples of proportion. Such questions, and those of the inverse type *If tennis balls cost* 32*s. per dozen, find the cost of each* can be treated as mere multiplication and division. Thus ratio, rate and proportion are used informally at a very early stage of the arithmetic course. The compiling of a ready reckoner, showing perhaps the cost of 1, 2, 3, 6, 12, 24, 36 tennis balls, implants the idea of the relationship between two variables. The ready reckoner also provides an informal introduction to the idea of ratio, for in the course of its construction it will be at once evident that the easiest way to find the cost of 24 balls is to double the cost of 12; and, when the table has been completed, it can be pointed out that 36 is 3 times 12, or 1½ times 24, and that the table can be checked at any point by comparing one reading with another in this way.

In both the examples of the last paragraph the rate is assumed to be uniform and a straight line graph appears if the readings in the table are plotted on squared paper; but it is undesirable that all the graphs in the early stages should be linear. An example of a non-linear graph, for which the data can be collected by the pupils, is the flow of traffic in vehicles per hour. Graphs of this type help to bring out the important fact that rates need not be uniform. In contrast the straight line graph comes to be recognised as the pattern of a constant rate of growth or change, and its slope is seen to give a measure of the rate. The graph showing the number of apples given to 1, 2, 3, 4, . . . children at the rate of 3 apples per child is similar to that showing the number of gallons of water which have flowed from a tap in 1, 2, 3, 4, . . . minutes at the rate of 3 gallons per minute; that the latter may be used to give intermediate values, while the former cannot, is a simple illustration of the important ideas of continuity and discontinuity.

Questions of the type

If 6 *eggs cost* 1*s.* 9*d. find the cost of a dozen.*

If 100 *identical castings weigh a ton, what would* 25 *weigh?*

may be taken at this stage, as well as some in which it is natural and proper to deal with one unit as a first step, e.g.

If a motor coach is hired for 36 *people at a cost of* £9, *find the cost per head and the charges for groups of* 5, 6 *and* 8.

If an aircraft covers 200 *miles in* 25 *minutes, find its speed in miles a minute, the distance it would travel in* 5 *seconds, and the time it would take for* 5 *miles.*

It will be noticed however that, in the simple examples so far mentioned, there is no difference in method, or in degree of difficulty, between those in which the unit step is used and those which are done directly. The process is simply a multiplication or a division.

After this, perhaps at a distinctly later stage, a more formal treatment of ratio and proportion will be undertaken and the appropriate technical terms introduced. The teacher's first task will now be to make the idea of ratio explicit: for this, visual aids may well be used. Two lines on the blackboard can be compared roughly by ratio without their actual lengths being known,

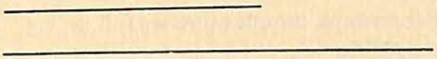


FIG. 8.2

Heights, preferably distant heights, can be compared. The proportions of the human body can be studied, (in Fig. 8.3, the man's height is h units), and it is worth while seeking out examples of ratio in which the actual measurements

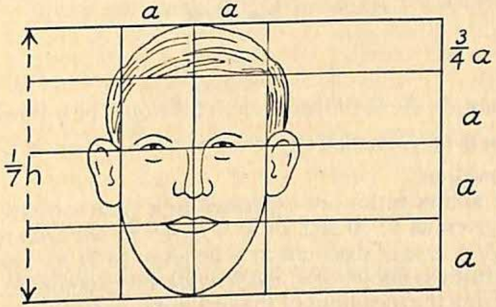


FIG. 8.3

are not known, e.g. the apparent distances between stars, the length and breadth of Australia, the length and height of a locomotive seen in a picture. These rough comparisons will be followed by more accurate ones in which the actual measurements necessarily enter, and the ratio notation will now be introduced. The statement '$A : B = 3 : 4$' means that if A is divided into 3 equal parts then B can be divided into 4 parts of the same size.

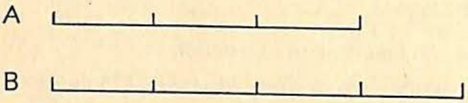


FIG. 8.4

Hence it implies a two-way relationship: A is $\frac{3}{4}$ of B, and B is $\frac{4}{3}$ of A. A similar statement for a three-term ratio, such as '$a:b:c=3:4:5$' (for the sides of a triangle, for example)

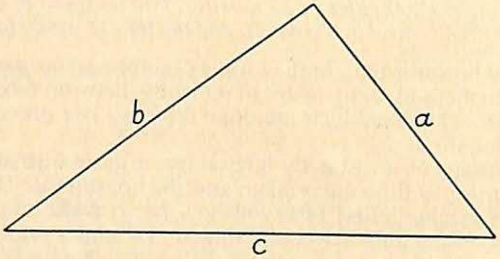


FIG. 8.5

implies six such relationships. Simple exercises follow, e.g.

If $b=20$ *cm, find a and c.*

These may be done informally, with the aid of sketch diagrams. Such a question as

Divide 24 *cm in the ratio* 3 : 4 : 5.

may be taken at this stage, a diagram being used to show

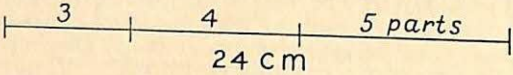


FIG. 8.6

that the parts are $\frac{3}{12}$, $\frac{4}{12}$, $\frac{5}{12}$ of the whole. This would be followed by

Divide 20 *cm in the ratio* 3 : 4 : 5.

and similar questions.

In practical affairs ratios are expressed in a great variety of ways. A map scale may be given as 1 : 10 560, or as 6 *inches to one statute mile*, or as 880 *feet to one inch*. A rate of discount may be given as 7½ *per cent* or as 1*s*. 6*d*. *in the £*. Such phrases as *one-third as much again* are in common use. Time is well spent on clarifying the meanings of these alternative forms.

Reference should now again be made to the ready reckoner, and examples in proportion should be given. These should begin with whole-number multipliers and proceed to easy fractions as, for example, in

If it takes 40 *minutes to fold* 1000 *pamphlets, how long would it take to fold* 2000*? How long to fold* 1500*?*

If 150 *hymn-books cost* £9 7*s*. 6*d*. *find the cost of* 100.

If a man walks 4 *miles in an hour, how far will he walk in* 40 *minutes?*

It is unwise to impose the fractional method on pupils who cannot use it easily. It is better to let them pass through the intermediate stage of finding the cost of 50 hymn-books or even of one. Later they may invent the fractional method themselves as a short cut. There are of course many examples in which it is of interest to know the cost of one article, and for these the unitary method is appropriate. It should not be developed into a routine. The compiling of a ready reckoner, in which the cost of one article will naturally be shown, brings out also the fact that it is easy to pass from one reading to another without going through the 'unit step' each time. The writing out of questions in proportion is a matter in which pupils need varying degrees of help. First they must be induced to write a sentence stating the fact given; then a similar sentence, with a blank in it, for the fact to be found. This form, because it sets out the comparison between the two cases, is instructive and helpful. It appears sooner or later that it is advisable to look ahead when composing the first of these sentences and to arrange it so that the unknown quantity will be at the end of the second sentence.

The idea of matching two sets of numbers represented by points on two straight lines can be useful with pupils who have difficulty with the English.

If 20 *oranges cost* 4 *shillings, find the cost of* 15 *oranges.*

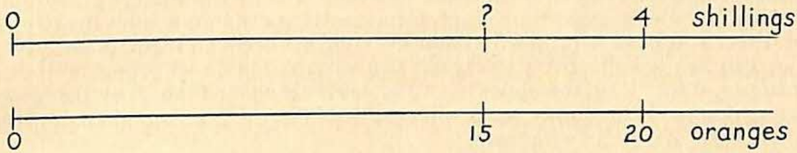


FIG. 8.7

We have a choice of two methods corresponding to the two methods of comparing the four intervals:

(i) The number of oranges changes in the ratio $\frac{15}{20}$. Therefore the required price is

$$4 \times \frac{15}{20} \text{ shillings} = \dots .$$

(ii) We pay at the rate of $\frac{4}{20}$ shillings per orange. Therefore the cost of 15 oranges is

$$15 \times \frac{4}{20} \text{ shillings} = \dots .$$

The first method uses ratio, and the second rate. In the Unitary Method, we say

$$\text{20 oranges cost 4s.,}$$

$$\therefore \quad \text{1 costs } \frac{4}{20}\text{s.,}$$

$$\therefore \quad \text{15 cost } 15 \times \frac{4}{20}\text{s.}$$

This uses ratio, because we are comparing the number of oranges, but the rate appears too, in the intermediate step.

In general the more complicated examples should be avoided. Questions in which fractional quantities have to be compared, e.g.

If $1\frac{1}{2}$ *yards of material cost 23s. 9d. what would* $2\frac{1}{3}$ *yards cost?*

may be treated by reducing to a smaller unit (e.g. 6 inches).

Some attention should be given to problems in which the quantities are not in direct proportion, e.g.

If it takes 4 minutes to boil an egg, how long does it take to boil two eggs?

If a pound of butter lasts 3 people for a week, how long will it last 6 people?

If a man has £200 when he is 20, how much will he have when he is 60?

If a halfpenny weighs $\frac{1}{5}$ *oz. how much does a penny weigh?* ($\frac{1}{3}$ *oz.*)

If a boy walks to school in half an hour and can cycle three times as fast, how long does he take to cycle?

If a metal plate 3 inches square weighs 6 oz. what would be the weight of one of the same thickness 6 inches square?

It is at this stage that the terms *proportion*, *direct proportion* and *inverse proportion* are needed. Further tables should be compiled (on the ready reckoner basis) and graphs drawn from them. A money-quantity graph for a constant buying rate and a speed-time graph for a constant distance exhibit the nature of direct and inverse proportion and the contrast between them. A slide rule can also be used effectively to exhibit two variables in direct proportion. For instance, if the '1' on the upper of two scales is set against the '2' on the lower scale, points on the upper scale will read half the value of the points on the lower scale in contact with them.

Some correlation should be established with scale drawing and work on similar triangles. The introduction of trigonometry is thus assisted and trigonometry can be used in turn to illustrate and strengthen the knowledge of ratio already acquired. Some classes may be able at this stage to tackle the law of squares for areas of similar figures, followed by that of cubes for volumes.

A ready reckoner or a table for a direct proportion graph may be extended in either of two ways; we can proceed from one reading to another by ratio (multiplying the number of articles and the cost by the same number); or we can find the new reading by knowing the rate (multiplying the cost per article by the number of articles). For inverse proportion (e.g. the times taken for a journey at different speeds) there is a similar choice: we can alter the two quantities in inverse ratio (multiplying one and dividing the other by the same number); or we can use the fact that the product of the two quantities is constant. The interpretation of this product in particular examples is of some interest: it may in one case be the length of the journey, in another the measure of a piece of work in man-hours. This may lead up to the laws $y=kx$, and $xy=K$ or $y=\frac{K}{x}$, and these should be introduced or illustrated by graphs.

Such compound units as *man-hours* are useful for questions of the type

If a job can be done by 12 men in 7 working days, how much longer will it take if 5 of them stop work after the first 4 days?

Other good examples of these product-units are the 'ration' (e.g. one man's food for a week) and the 'unit' of electricity (defined as a kilowatt-hour). Questions in compound proportion are apt to be artificial and should not be made too elaborate. In commercial practice such computations are often avoided or simplified by the use of tables; the construction and use of such a table can sometimes be more illuminating and interesting than working a series of miscellaneous examples.

The idea of increasing or decreasing in a given ratio may well be introduced in connection with ready reckoner tables or with graphs, and should be thought of at first simply as multiplying by a fraction greater or less than 1. The phrase *increase in the ratio* 5 *to* 4 may be confusing to beginners and can be replaced at first by *increase in the* ratio $\frac{5}{4}$. The use of ratios and multipliers is more than a piece of arithmetical technique. The habit of thinking in such terms enlarges the mathematical outlook, carrying the pupil forward to trigonometry, to variation and to the idea of functionality. The teaching of ratio in arithmetic should be directed to this end.

Once pupils have acquired an understanding of and a facility in the use of ratio and rate, the contexts in which they are required to apply the ideas can be enlarged. These contexts will depend on the pupils' vocabulary and experience; often their success will depend as much on their full understanding of the various terms used in a particular sphere or topic,—e.g., the rateable value of a property or the percentage yield on an investment—as on their ability to carry out the calculations involved. The teacher's role is to encourage this full understanding by providing numerous questions involving the terms and requiring only simple calculations; a few questions taken from the adult business world are likely to stimulate interest, but too many may prove time-wasting.

Moving from one topic to another enlarges the vocabulary of the pupils, and increases the range of problems they can tackle without necessarily demanding a different arithmetical process. Generally the principle of direct proportion is used.

Among topics introduced may be some of the following; the order in which they are placed is not necessarily the order in which they should be taken.

Discount.
Simple interest.
Rate of profit or loss.
Depreciation in value.
Compound interest.
Income tax.
Hire purchase.
Local Authority finance.
Insurance.
Investments (shares and stocks).
Pressure.
Speed.
Density.
Household arithmetic.
Monetary exchange.

The problems which arise may be 'direct' or 'inverse'; the latter are the more difficult, and some guidance in the method of attack is called for. As stated earlier in this chapter, the writing of a statement in a particular way is often the key; the following outlines of worked examples serve as illustrations. They should *not* be taught to pupils as methods which they must use; but rather, each pupil should be encouraged to develop his own style subject to the requirements of orderly presentation and accurate calculation.

Simple interest

Direct: Find the interest on £350 for 7 months at 4% p.a.

Two possible approaches to the solution are:

(i) Interest on £100 for 1 year $= £4$.

$\therefore$ interest on £350 for 7 months $= £4 \times \frac{350}{100} \times \frac{7}{12}$.

(ii) Interest on £350 for 1 year $= \frac{4}{100} \times £350$, etc.

Inverse: What sum of money invested at 4% p.a. amounts to £358 3s. 4d. in 7 months?

The statement needs to end with the sum invested so it will help to know the result of investing £100.

Interest on £100 for 7 months $= £4 \times \frac{7}{12}$,

$= £2\frac{1}{3}$.

$\therefore$ £102$\frac{1}{3}$ is the amount when the principal is £100.

$\therefore$ £358$\frac{1}{6}$ is the amount when the principal is £100 $\times \frac{358\frac{1}{6}}{102\frac{1}{3}}$, etc.

Some teachers may prefer to use algebra in which case the question may be answered as follows:

Let the sum invested be £x.

Interest on £x at 4% for 7 months $= £x \times \frac{4}{100} \times \frac{7}{12}$,

$$= £\frac{7x}{300}.$$

$$\therefore\ x + \frac{7x}{300} = 358\tfrac{1}{6}.$$

$$\therefore\ \frac{307}{300}x = \frac{2149}{6}.$$

$$\therefore\ x = \frac{7 \times 300}{6},$$

$$= 350.$$

$\therefore$ the sum to be invested $= £350$.

Profit and loss

In the direct case it is often better to calculate the gain (or loss) and then to add (or subtract) than to use the factor $\left(1 + \frac{r}{100}\right)$. E.g. if the question were

An article bought for £7 18s. 4d. was sold at a loss of 3%; what was it sold for?

it would be better to calculate 3% of £7 18s. 4d. and to subtract than to find 97% of £7 18s. 4d.

Inverse: By selling sugar at 9d. a lb a grocer gained 26% of the purchase price. What did 1 cwt of sugar cost him?

He sold 1 cwt of sugar for $\frac{3}{4} \times 112$s. i.e. for 84s.
He gained 26% of cost price.

$$\therefore \text{ selling price } = \frac{126}{100} \text{ of cost price.}$$

$$\therefore \text{ cost price } = \frac{100}{126} \text{ of selling price.}$$

$$\therefore \text{ cost price of 1 cwt } = \frac{100}{126} \text{ of 84s.} = \text{66s. 8d.}$$

Depreciation

Suppose that the value of a new car depreciates by one-sixth of its value during its first year, and in subsequent years by one-eighth of the value at the beginning of the year.

Direct: A new car cost £1200; *find its value after* 2 *years.*

$$\text{Value after 1 year } = £1200 \times \frac{5}{6};$$

$$\therefore \text{ value after 2 years} = £1200 \times \frac{5}{6} \times \frac{7}{8} = £875.$$

Inverse: The value of a car after 2 *years is* £875; *find its original value.*

$$\text{Value after 1 year } = £875 \times \frac{8}{7}.$$

$$\therefore \text{ original value } = £875 \times \frac{8}{7} \times \frac{6}{5} = £1200.$$

Familiarity in forming the necessary mutlipliers, particularly in the inverse cases, is important.

Compound interest

Compound interest operates when the account is made up at regular intervals in such a way that the interest as it becomes due is added to the principal. This is the case in the calculation of mortgages and savings accounts. In actuarial theory, as applied to annuities and life assurances, the money invested is regarded as being increased continuously in such a way as to result in a certain annual percentage increase.

Such questions as *Find the compound interest on* £395 *for* $3\frac{1}{2}$ *years at* 4 *per cent* cannot be answered without instructions as to how the account is made up. In practice, if the account is made up annually, interest might not be earned for the last half-year. If the account is made up half-yearly by the addition of 2 per cent, the question should be framed accordingly. (The effective annual rate is then 4·04 per cent.) The theory of continuous growth gives the amount as $£395 \times 1{\cdot}04^{3\cdot5}$.

Questions in compound interest provide exercise in:

1. straightforward computation, using a compact arrangement of working;
2. logarithms, using the formula $A = PR^t$, where $R = 1 + \frac{r}{100}$;
3. making and using tables, including those used for annuities, etc.

Local Authority finance: Rates

The pupil should not only learn that local 'rates' are paid at so much in the pound on the 'rateable value' of each house, but should also look at this payment from the receiving end, that of the local authority. Here are two examples which illustrate the type of question which may be given to the pupils.

Example 1. *The rateable value of a house is* £56; *how much must the householder pay each half year, if the rate is fixed at* 17*s.* 2*d. in the* £ *per annum?*

Example 2. *A new village hall estimated to cost* £8700 *is to be built. Compensation and a ministry grant will provide* £6200 *of the money needed and the rest is to be paid for by an extra rate of* 2*d. in the* £1. *What is the total rateable value of the village?*

If the actual cost of the hall increases to £9950 *and the extra cost is borne entirely by the rates, how much extra will a certain villager pay if the rateable value of his property is* £60?

Speed

Example 1. *A cyclist takes* 15 *minutes to get to school when riding at an average speed of* 10 *m.p.h. On the return journey by the same route the average speed is reduced to* 6 *m.p.h. What is the average speed for the double journey?*

Average speed $= \dfrac{\text{total distance}}{\text{total time}}$. (Units: miles, hours)

Distance from school $= 10 \times \dfrac{1}{4}$ miles $= \dfrac{5}{2}$ miles.

Time for outward journey at 10 m.p.h. $= \dfrac{1}{4}$ h.

$\therefore$ time for return journey at 6 m.p.h. $= \dfrac{10}{6} \times \dfrac{1}{4}$ h $= \dfrac{5}{12}$ h.

$\therefore$ average speed $= \dfrac{5}{\frac{1}{4} + \frac{5}{12}}$ m.p.h.,

$= \dfrac{5 \times 12}{3 + 5}$ m.p.h.,

$= 7\frac{1}{2}$ m.p.h.

Example 2. *If the maximum speed of a car* 16 *feet long is* 70 *m.p.h., what is the least time it will take to pass a transport lorry* 30 *feet long moving in a parallel traffic lane at* 45 *m.p.h. and how far will it travel while doing so?*

Rate of overtaking $= (70 - 45)$ m.p.h. $= 25$ m.p.h.

$= \dfrac{25 \times 88}{60}$ ft per sec,

$= \dfrac{5 \times 22}{3}$ ft per sec.

Distance to be gained $= (16 + 30)$ ft $= 46$ ft.

Time taken to pass $= \dfrac{\text{distance}}{\text{speed}}$,

$= \dfrac{46 \times 3}{5 \times 22}$ sec,

$$=\frac{69}{55}\text{ sec,}$$

$$=1{\cdot}3\text{ sec,}$$

correct to nearest $\frac{1}{10}$ sec.

$$\begin{array}{r} 1{\cdot}3 \\ 55\,\overline{)\,69} \\ 55 \\ \hline 140 \\ \underline{165} \end{array}$$

Distance moved in this time = speed × time,

$$=70\times\frac{88}{60}\times\frac{69}{55}\text{ ft,}$$

$$=129\text{ ft, correct to nearest ft.}$$

Variable rates

Speeds are, in practice, seldom uniform. We therefore are concerned in elementary stages with *average* speeds. A further important development is the consideration of *variable* speeds and, in general, variable rates. The pupils should realise that a steeper slope on a distance-time graph implies a greater speed and, in the later stages of the arithmetic course, the estimation of the speed at an instant by drawing a tangent helps to prepare the way for the calculus. This is treated more fully in the Association's Report *The teaching of calculus in schools.*

A graph of the Amount at compound interest at, say, 10% is also instructive: while the rate of interest is constant, and there is a constant growth factor of 1·1, the rate of growth of the Amount increases with time. This graph reappears in an algebraic setting at a later date.

CHAPTER 9

MENSURATION

MENSURATION has been described as one half of arithmetic. It is the application of arithmetic to geometrical aspects of concrete things which are easily open to observation. Its successful pursuit involves geometrical perception developed by such experiences as drawing, measuring and modelling, and by observation. The extent of actual practical exercises at the secondary stage depends on the experience of pupils in the primary years, but they can seldom be omitted altogether.

It is essential that clear conceptions of units of measurement should be in the pupil's mind and that the units stressed are those in common use. Some account of the history of the commoner English measures, e.g. the furlong, the acre, is usually of interest. Metric as well as English measures should be included, also some metric equivalents of English measures, for instance, 1 inch = 2·54 cm, 1 metre ≃ 39·37 in., 1 km ≃ $\frac{5}{8}$mile.

Many of the geometrical concepts relating to areas and volumes are likely to have been accepted from experience before the secondary stage. But difficulties in applying arithmetic to measured lengths and shapes will arise, as, for example, in considering the area of curved surfaces, the relations between volume and capacity, the possible equality of surfaces or volumes of differing shapes. These all require geometrical as well as numerical treatment.

Length. Pupils at the secondary stage have probably had experience of measuring lengths and of the commoner units employed for this purpose, but the range of this experience will vary greatly. Many of the suggestions contained in the following notes may have received some preliminary treatment, but adequate appreciation of the considerations raised cannot be assumed.

(i) Estimation of a length may be treated as a first crude measure of it. The individual should carry in his mind useful standards for comparison in estimating heights and distances. Thus a height may be estimated as 'About 3 times the height of the door, i.e. about 20 ft', or the distance of an object may be estimated as 'Twice the length of a cricket pitch, or between 40 and 50 yd'. For the shorter standards, multiples up to perhaps six times may be used, but if the distance is a long one, a near-equivalent or a longer standard may be used, for example 'A few yards further than the distance from the school door to the school gate', or 'Rather more than half the length of a netball court'. Thus the use of estimation brings out clearly the notion of ratio implicit in all measurement.

(ii) The application of individual measures such as the pace, the foot-length, the span and the finger-joint may serve as a second approximation. Interest may be aroused in bodily proportions, for instance in the usual equality of height and arm span. The finding by an individual of his average pace is valuable both as a practical exercise and as an established result for future use.

(iii) Longer distances are more difficult to estimate and to measure than short ones. Numerical relationships such as 1760 yards to the mile may have little meaning in terms of reality even when related to some known measured mile. The techniques of measuring distances beyond the limits of the school buildings are interesting and mathematically informative.

(iv) A curved length is more difficult to measure than a straight segment of a line. The use of a variety of measuring instruments for finding a curved length, e.g. a flexible string or tape-measure and a rolling wheel, provides useful data for discussing the approximate nature of measuring.

(v) All measuring is approximate. The round numbers given in text-book examples mask this, for they are idealised or hypothetical measures, comparable with the suppositions made in geometry, such as '*AB* is a straight line', or with the simplifications used in all branches of science, e.g. 'uniform density', 'rigid body', 'pure oxygen'. But round numbers may also be approximations and then it becomes important to know whether '13 ft' means '13 ft to the nearest foot' or '13·0 ft' or '13·00 ft.' This is not easy for a pupil to appreciate unless it is carefully related to his experience of measurement. If measuring is confined to man-made objects, such as tables and rooms, numbers which appear to be round numbers are likely to occur. These reflect the maker's skill in working to a specification and they provide an opportunity for such questions as 'How close is it?' 'Do you mean "18·0 inches" or "18·00 inches"?' On the other hand the collection of natural statistics such as pupils' heights, the heights of plants, or the measuring of outdoor distances, will more obviously demand that pupils should work to some chosen degree of accuracy.

In the collection of statistics results may have to be given in categories, for example heights of 5 ft and less than 5 ft 1 in., 5 ft 1 in. and less than 5 ft 2 in. etc. Individual results should always be stated in some form related to the degree of accuracy required, such as 'correct to the nearest inch' or 'within $\frac{1}{16}$ inch'.

(vi) In general 1 or 2 per cent accuracy is a reasonable standard for school measuring. This standard is given by the ratio of allowable error to the measure itself. Correct 'significant figures' are therefore important and correct decimal places are not necessarily a guide to what is relevant or possible.

(vii) The degree of accuracy of a measure depends both on the instrument used and on the user. Parts of a unit expressed in decimal fractions conveniently exhibit degree of accuracy. Thus measuring in inches and tenths of an inch, and estimating tenths of a tenth, is an activity appropriate to the learning of decimal fractions.

(viii) Consistency is not to be expected in the results of a number of independent trial measurements of a length. What is important is that inconsistencies should not exceed reasonable tolerances.

(ix) The value of improving standards of accuracy by refinements in the instruments used may at later stages be discussed in relation to the purpose of the work. This kind of consideration may arise in the science room, where calculations may include measured results in other fields in which equal standards of accuracy are not possible.

(x) To know the standards of accuracy used in measuring lengths is important when dealing with the mensuration of areas and volumes.

Area. Discussion of the concept of area and its measurement has probably been begun in the primary school, but revision of first principles is recommended. It is a novel idea to have to choose both the size and the shape of the unit of measurement. The convenience of the square as an intermediary in the comparison of two areas of different shapes is exhibited, for instance, in the square grid on a large-scale map (though this is mainly used for fixing position). The fact that a square inch of area is not necessarily of square shape, i.e. an inch square, has to be noted.

The determination in square units of any enclosed area, e.g. the surface of a leaf, by counting squares and part squares on a superimposed grid is an interesting exercise in itself and a useful approach to revision of any rules already learned for finding the areas of particular shapes, including the rectangle.

Transparent squared paper, obtainable in various rulings, may be used as a grid. Shapes can be duplicated in various ways, or found in atlases and other books so that several pupils may make and compare estimates of the same area. Shifting the grid to a new position on the shape to be measured gives a new count of squares and part squares. (An alternative is to use squared paper and to duplicate shapes on to it. This can be done with a spirit duplicator or with a map roller which can often be borrowed from the geography department.) Estimates of part squares can be made in various ways, perhaps beginning with counts of parts more and less than a half.

The fundamental process of square counting, even when applied to the simple case of the rectangle, is worth establishing as a mental picture available in all cases of difficulty. Moreover, the area of a rectangle provides a diagrammatic representation of multiplication. A series of rectangles, drawn to scale if necessary, such as:

22 ft by 25 ft
3 in. by $2\frac{1}{2}$ in.
$3\frac{1}{4}$ in. by $2\frac{1}{2}$ in. (Fig. 9.1)
2·3 in. by 0·3 in. (Fig. 9.2)

can be used to illustrate the rules appropriate for multiplication of integers and of vulgar and decimal fractions.*

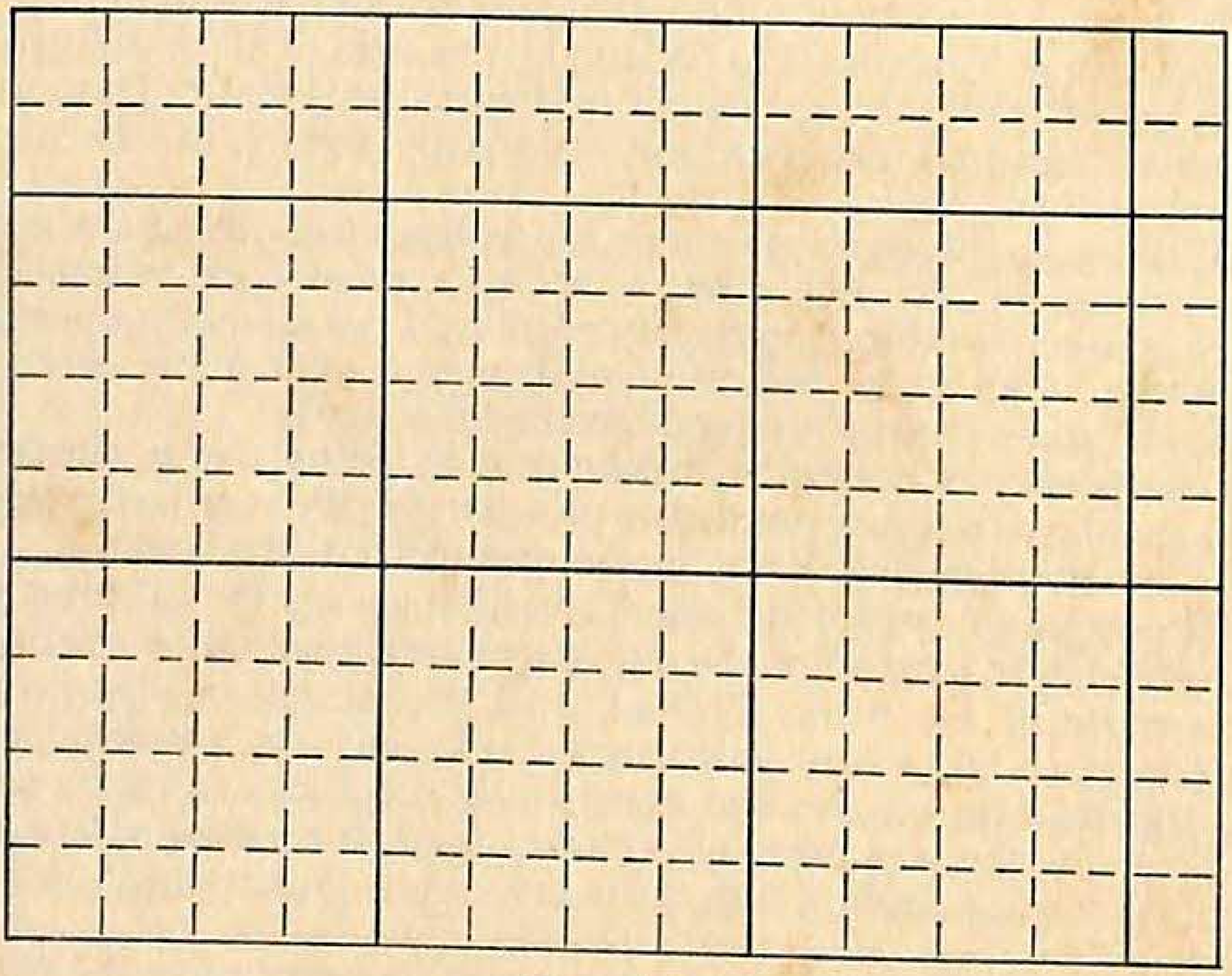


FIG. 9.1

---

* See *Mathematics in secondary modern schools*, 4.812, 6.246, 6.248, also Chapter 6 of this Report.

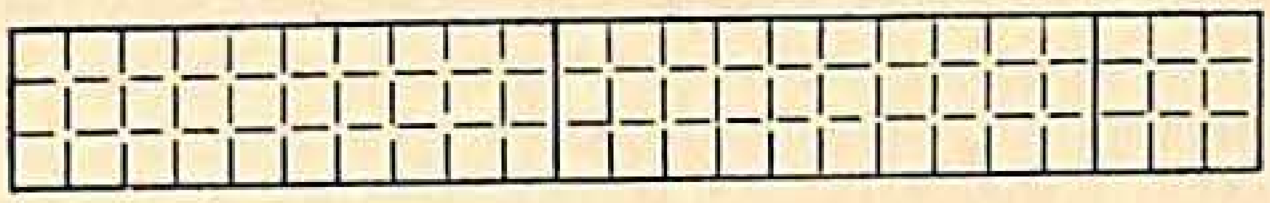


FIG. 9.2

Difficulties with units which often occur when area is treated purely arithmetically are resolved by an appeal to a squared diagram. For instance, an area which is 2 feet by 5 inches must be treated either as

24 inches by 5 inches with an answer in square inches

or 2 feet by $\frac{5}{12}$ feet with an answer in square feet.

If the mistake is made of saying 2 × 5 units, the units are not squares but rectangles 1 foot by 1 inch.

Further points to be noted are:

(i) Numerical and graphical work on the areas of squares with their direct bearing on tables of square measure should be included.

Thus a square of side 3 feet contains 3×3 square feet;
a square of side 1 metre contains 100×100 square centimetres;
a square of side 1 inch contains 4×4 squares of side $\frac{1}{4}$ inch,
or a square of side $\frac{1}{4}$ inch is $\frac{1}{16}$ square inch.

(ii) The expression '120 yards by 80 yards' is commonly written '120 yards × 80 yards'; also, for instance, a carpet may be marked '8 ft × 12 ft'. In this context the symbol ' × ' stands for the word^6 by' and not for multiplication, and it is better not to use '129 yards × 80 yards' in statements of steps in calculation. Confusion may be avoided by making a practice of stating units used before beginning calculations. For instance, the answer to the question, 'What sized squares shall we use?' might give rise to the following form of statement:

Find the area of a piece of wood 3 *feet by* 5 *inches.*

Length $= 3$ ft. (Working in feet and square feet.)

Breadth $= \dfrac{5}{12}$ ft.

Area $= 3 \times \dfrac{5}{12}$ sq. ft $= 1\frac{1}{4}$ sq. ft.

(iii) From the use of a squared grid for finding the area of any shape the idea readily emerges that a similar shape with double the linear dimensions has four times the area, and in general that areas of similar shapes 'grow' by squaring the ratio of their sides.

(iv) There is a well-known class of problems about the areas of paths bordering lawns, the volume of wood required to make boxes, the weight of metal in pipes, etc. There is no particular difficulty in the method based on subtracting from the whole the area, or areas, not required, but considerable resistance to it is encountered in the early stages of the secondary school from pupils who have been accustomed to dividing the border of a rectangle into a number of rectangles. Diagrams may not give enough help to pupils who are still having difficulty with the idea of subtraction of areas. Pieces of card which can be moved about, superimposed on one another in different positions and, if necessary, cut up may help to make things clear. The result of cutting out a corner piece from a rectangle can be compared with the result of cutting out an equivalent piece from inside.

(v) From the area of a rectangle are derived successively rules for the area of the parallelogram, triangle and trapezium. It should be emphasised that any side of a parallelogram or triangle may be taken as 'base'. Particular attention should be paid to those figures in which the altitude falls outside the base. It is a good exercise to calculate the altitude to the hypotenuse of a right-angled triangle by equating the area of the triangle found by using the two sides containing the right angle as base and height to that of the triangle taking the hypotenuse as base. For the area of a trapezium pupils should eventually derive the formula $\frac{h(a+b)}{2}$ and appreciate that its use shortens computation.

(vi) The mensuration of the circle is considered later.

Volume. The fundamental difficulties in dealing with measurement of volume are geometrical rather than arithmetical. It is essential that each individual should hold clearly a few basic ideas.

(i) The unit of volume is a cube with side of unit length. Cubic units may be built up into rectangular solids and the number of cubes in the solid can be considered as the number of cubes standing on one face multiplied by the number of such layers.

This may be reduced to the continued product of the numbers of cubes along a length, a breadth and a height.

(ii) If the numerical difficulties in dealing with areas, e.g. in areas involving fractions of a unit, have been dealt with thoroughly, a very simple treatment of analogous difficulties in volume finding should suffice.

(iii) A cubic yard contains 3 × 3 × 3 cubic feet. A box of this size and shape holds a surprising amount: over 168 gallons or about a ton of gravel.

(iv) Units of volume may be related to units of capacity found by filling one vessel with the contents of another. Thus 1 litre = 1000 cubic centimetres. A cubic foot holds about 6¼ gallons, etc.

(v) Capacities are difficult to estimate and to measure directly. One gallon is legally defined as the volume of 10 lb of water. (See Appendix.) Another useful equivalent is that one cubic foot of water weighs about 1000 oz.

(vi) It is strongly recommended that actual blocks are used to illustrate inch cubes; these greatly add to the reality of the discussion and can be made by sawing up wood of square cross-section, alternatively the pupils can make cubes in connection with geometry and these can be stored for use in arithmetic. (For an interesting method of constructing a cube, and other polyhedra, see an article by A. R. Pargeter. Plaited Polyhedra, *The Mathematical Gazette* Vol. XLIII, No. 344, May 1959.) Work on volumes should be linked up with elementary work in geometry on plans and elevations. In particular, the idea of cross-section needs to be carefully brought out.

(vii) The idea that volumes of similar solids are compared by the cubes of their linear dimensions follows by analogy with areas (see p. 47) by replacing the squared grid by a formation of cubes. Again, the comparison of the surfaces of similar solids by the squares of their linear dimensions is implicit in what was said on p. 47. The importance of these two ideas taken together is capable of vivid practical illustration, particularly in the field of biology. (See *Teaching mathematics in secondary schools*, Ministry of Education Pamphlet 36, p. 114–116.)

Circles and cylinders. (i) The circumference of a circle. The idea of the length of a curve is not a difficult one for the pupil who is used to the measuring tape. Body measurements of chest expansion, waist, etc., are common enough and the natural way for a child to measure the circumference of a circle is to wrap a tape or some cotton round it. The first point to make is that the circumference is proportional to the radius. The pupils should be brought to see that one circle is an enlargement of another. The circumference of circles of different radii may be estimated by stepping off in small steps with compasses, by winding a piece of cotton round a cylinder a number of times (ten is a good number) and by measuring the distance rolled by a wheel in a number of revolutions. When a number of circumferences have been measured, the results can be compared on a graph of circumference against radius, or the ratios may be found and compared.

It needs to be stressed from the beginning that π is the ratio of the circumference of a circle to its diameter. Its value is 3·141 592 65 ..., and there are a number of convenient approximations: $3\frac{1}{7}$ is very slightly nearer than 3·14, but not as near as 3·142. The first use of π is an excellent opportunity to bring in a little of the history of mathematics.

(ii) Any approach to the area of a circle should begin with estimating the areas of circles of different radii by counting squares. (See p. 46.) With less able pupils the formula for the area will be found from these data. But with abler pupils the formula for the area may be deduced from the formula for the circumference as follows.

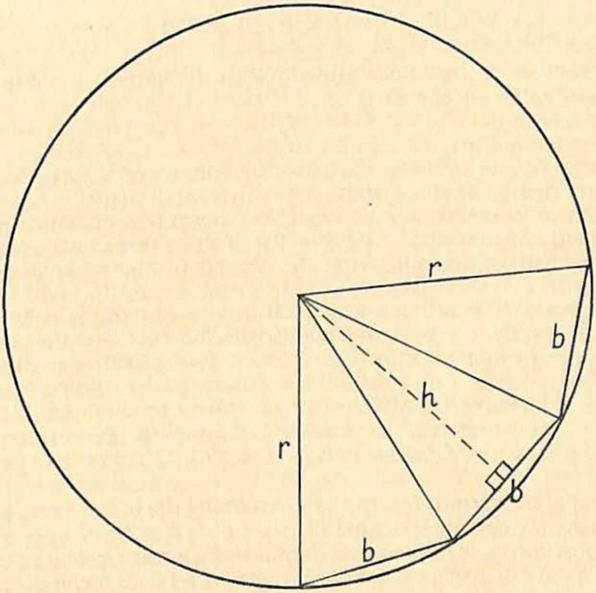


FIG. 9.3

A regular polygon of n sides is inscribed in a circle of radius r and the centre of the circle is joined to the vertices of the polygon (Fig. 9.3). The polygon is thus divided into n congruent isosceles triangles whose bases are the sides of the polygon. Let the sides of the polygon be of length b and their distance from the centre be h. Then the area of each triangle is $\frac{1}{2}bh$ and so the area of the polygon is $\frac{1}{2}nbh$, that is,

$$\text{the area of the polygon} = \tfrac{1}{2} \times (\text{perimeter of polygon}) \times h. \qquad (1)$$

Now note that, as the number of sides of the polygon is increased indefinitely,

(*a*) the area of the polygon approaches as close as we like to the area of the circle,
(*b*) the perimeter of the polygon approaches as close as we like to the circumference of the circle,
(*c*) h approaches as close as we like to r.

Thus the equation (1) gives us

$$\begin{aligned}\text{the area of the circle} &= \tfrac{1}{2} \times (\text{circumference of circle}) \times r,\\ &= \tfrac{1}{2} \times 2\pi r \times r = \pi r^2.\end{aligned}$$

At this stage of the course, the word 'limit' will not generally be used and the above argument will be strange to the pupils. It is thought better by some teachers to consider, as well, a regular polygon of n sides circumscribing the circle. In the same way it is found that the area of the circumscribing polygon approaches as close as we like to πr^2, so that the area of the circle lies between two areas which both approach as close as we like to πr^2.

(iii) The area of the curved surface of a cylinder is easily obtained once the formula for the circumference of a circle is known. If paper round a cylinder is unwrapped to form a rectangle, the pupils should be able to work out the formula for themselves.

(iv) The volume of a cylinder presents little difficulty if the pupils have a clear understanding of the area of a circle and the volume of a solid of uniform cross-section.

Pyramids and Cones. The formula for the volume of a pyramid is not an easy one to derive at the stage where it is usually first met. It may be assumed that pyramids on equal bases with equal heights are equal in volume and then a triangular prism is dissected, as in Fig. 9.4. If a cutter is available, cheese is a good material with which to illustrate the dissection. Many teachers, however, would prefer to construct models of card or paper and to build up a prism from three pyramids. Another approach is to assume that the volume is some fraction of the product of base area and height, in which case the constant $\frac{1}{3}$ is found most easily by considering the six pyramids obtained when the diagonals of a cube are drawn. A third method is to compare the volume of a pyramid and a prism with the same base and height, filling models of stiff cardboard with sand. A convenient size for models to be used is about three inches in height: they can be made during an earlier part of the course and kept for use at this stage.

For the cone, the volume follows by considering the limit (though this word will not usually be used at this stage) as the number of sides of a pyramid's base increases indefinitely. The area of the curved surface is better treated by considering how the surface may be laid out on a plane surface, becoming a sector of a circle.

Spheres. There is no harm in introducing the formulae for the surface area and volume of a sphere without justification, the proofs being deferred until the calculus is reached. The fundamental result connecting a zone of a sphere with the corresponding zone of the circumscribing cylinder may be given to

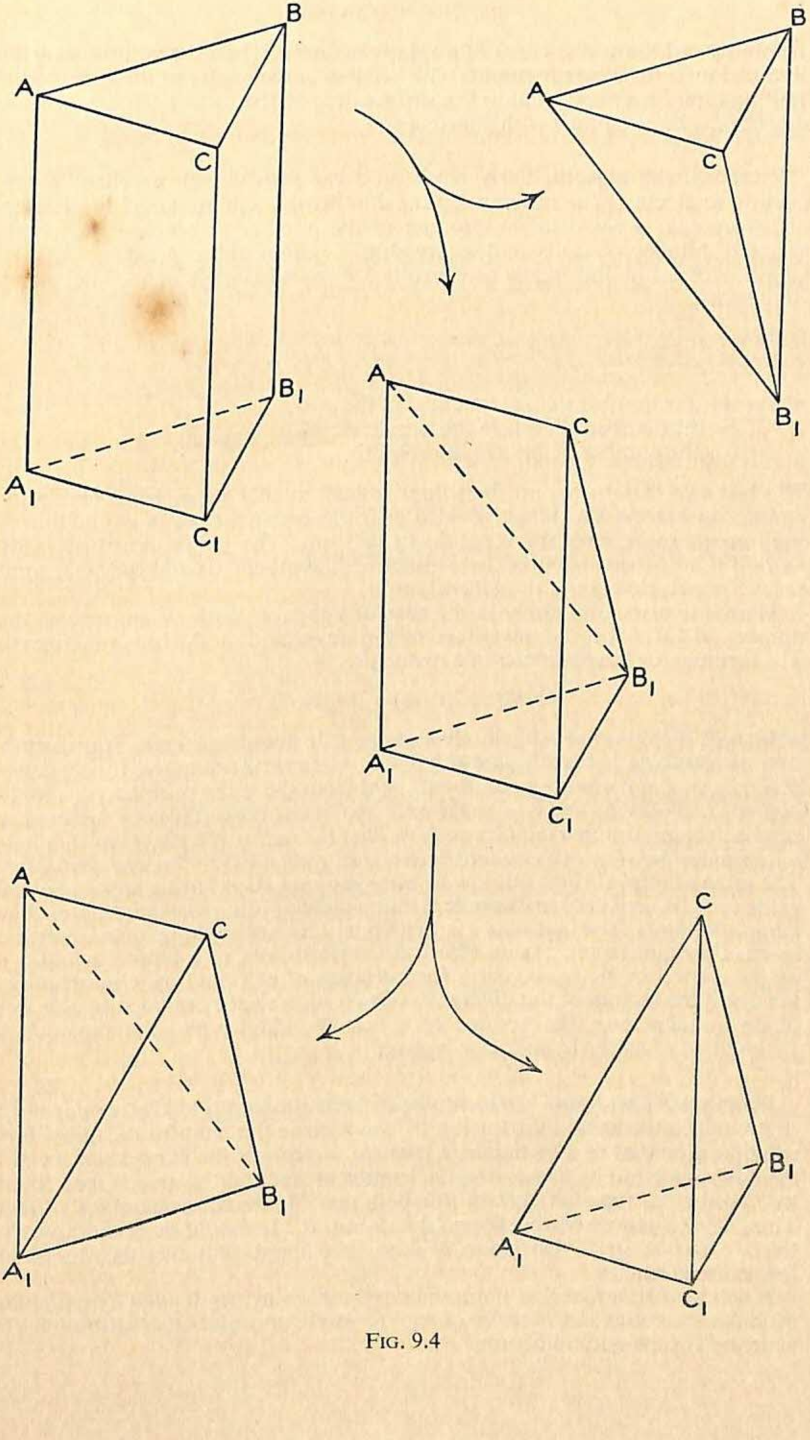


FIG. 9.4

the pupils and the surface area of a sphere deduced. Then the volume may be obtained by considering pyramids with vertices at the centre of the sphere and limiting total base area equal to the surface area of the sphere. This, however, should only be attempted with good classes.

Relations with algebra. Early work on areas provides an excellent introduction to algebra. The statement, 'The number of "square units" in the area of a rectangle is equal to the product of the number of units in the length and the number of units in the breadth,' is often abbreviated to 'Area = length × breadth' and pupils can readily see the advantage of the shorthand statements,

$$A=lb, \qquad l=\frac{A}{b}, \qquad b=\frac{A}{l},$$

where A is the number of 'square units' in the area,
l is the number of units in the length,
b is the number of units in the breadth.

The last two statements are best approached, at this stage, from first principles, considering a rectangle divided up into a number of squares and knowing how many squares there are down one side. The pupils can then easily work out how many squares there are down the other side and they will soon reach a conclusion about the general result.

The same principles apply in the case of volumes. With an increase in the number of variables, the advantage of the shorthand of algebra is increased. The formula for the perimeter of a rectangle,

$$P=2(l+b)=2l+2b,$$

written in these two forms, can give us the rule about brackets. Transformation of formulae is usually first done with numerical examples, for instance, *If* $S=2\pi rh$, *find* r *when* $S=25$, $h=10$, and from there the problem is generalised to, *If* $S=2\pi rh$, *find* r *in terms of* S *and* h.The advantages of factorising can be brought out in examples such as *Find the radius of a circle equal in area to the space between two concentric circles of radii eight inches and ten inches.* The same example brings out the advantage of not substituting any numerical value for π. A well chosen example is more likely to bring this point home than admonishments. The fact that π is written as a letter should help its treatment as an algebraic factor. Again, the total surface area of a closed cylinder is better treated in factorised form for purposes of calculations. Opportunities for using the factors of the difference of two squares are provided by the area of an annular ring, the volume of a hollow cylinder and the theorem of Pythagoras when the hypotenuse is given.

Units and Dimensions. The formulae of mensuration provide an opportunity to introduce the idea of dimensions. If we wish to find an area in square feet, whether it be that of a rectangle, a triangle, a circle or the curved surface of a cylinder, we begin by expressing the lengths in feet and the area is then found by multiplying together certain numbers two of which are numbers of feet. This may be expressed shortly as 'ft × ft = sq. ft'. It should be noted however that we do not, strictly speaking, multiply two lengths but only the number of feet in those lengths.

It will also be noted that if a number of cubic yards is divided by a number of yards the result is a number of square yards, and this observation may be expressed in the symbolic form

$$\frac{\text{cu. yd}}{\text{yd}} = \text{sq. yd.}$$

In the same way, in finding the time taken for a journey, we may divide the number of miles by the number of miles per hour, thus obtaining the number of hours taken. The 'dimensions check' for this is

$$\text{mi.} \div \frac{\text{mi.}}{\text{h}} = \text{h.}$$

Such ideas may be used as a check on the processes used in solving problems. The invaluable habit of thinking in terms of dimensions is at the same time instilled.

Consider, for example, the problem *Find the depth if a cubic foot of sand is spread over a rectangular area* 3 *yd long and* 30 *in. wide.* Some teachers would write the initial stages of the solution along the following lines:

$$\text{Length} = 9\text{ft.} \quad \text{Width} = \frac{30}{12}\text{ ft.}$$

$$\text{Depth} = \frac{\text{volume}}{\text{length} \times \text{breadth}} = \frac{1}{9 \times \frac{30}{12}} \text{ ft etc.}$$

Others like to make use of the check provided by dimensions and would write something like:

$$\text{Depth} = \frac{\text{volume}}{\text{length} \times \text{breadth}} = \frac{1}{9 \times \frac{30}{12}} \ \frac{\text{cu. ft}}{\text{ft} \times \text{ft}},$$

$$= \frac{12}{9 \times 30} \text{ ft, etc.}$$

Conversion to convenient units should be done in preliminary steps, as in the above example. More elaborate problems on change of units, such as

Find in francs per litre the price equivalent to 5s *per gallon if* £1 = 13·8 *fr and* 1 *litre* = 0·220 *gal*

may be treated as follows:

1 gal costs 5s.

(1 litre = 0·220 gal)

1 litre costs 0·220 × 5s.

$$(1\text{s.} = \frac{13{\cdot}8}{20}\text{ fr})$$

$$1 \text{ litre costs } 0{\cdot}220 \times 5 \times \frac{13{\cdot}8}{20}\text{ fr}, \quad \text{etc.}$$

Later on, some teachers may like to introduce the more compact form:

$$\text{Price} = \frac{5\text{s.}}{1\text{ gal}} = \frac{5 \times \frac{13{\cdot}8}{20}\text{ fr}}{\frac{1}{0{\cdot}220}\text{l.}}, \quad \text{etc.}$$

CHAPTER 10

GRAPHS

GRAPHS enter into every branch of school mathematics, as well as into such other subjects as physics and geography. For this reason they should not be regarded as a separate subject to be taught exhaustively at a certain stage, but should be brought in from time to time as occasion arises.

The purpose of any diagram is to convey information in a vivid manner. That of a graph is to exhibit the dependence of one variable on another. The teacher's aim, in graph work, should be to build up this idea of functional dependence, looking forward more to the calculus than to coordinate geometry.

A beginning may be made by examining ready-made diagrams such as can be found in newspapers, particularly in the advertisement columns. These will include pie-diagrams, isotypes and histograms, as well as ordinary graphs, and the observations made from them will at first be qualitative rather than quantitative. There is much to be said for the slogan 'Reading before writing; qualitative before quantitative'. The self-recording barometer provides a good example and rough graphs can be drawn on the blackboard to represent such things as speed of a motor-car, petrol-gauge readings, weight of a boy, rate of consumption of electricity during the course of a day. Pupils should be able to describe in simple terms what a graph tells them and may be encouraged to use their imagination in accounting for the ups and downs. They should also be able to sketch a rough graph on plain paper from such general information as 'the temperature rose steadily from 5 a.m. to 10 a.m., then more slowly, reaching a maximum at 12 noon, etc.'

Full understanding, particularly of a quantitative kind, will not come until pupils can draw graphs for themselves, and this should be begun at an early stage. Very simple examples should be chosen at first, based if possible on the pupils' own experience. Graph paper need not be used, but will soon be found to be convenient. Suitable early examples are:

The number of school periods allotted each week to the various subjects.
A pupil's own marks in a series of tests.
The number of members of the class whose birthdays fall in the different months of the year.
The height of a bean plant measured daily.
Takings in the school tuck-shop.
Daily temperature readings.

These may be represented by columns (i.e. by a series of 'vertical' lines on the paper) or, more conveniently, by points indicating the tops of the columns. The points (or the tops of the columns) may be joined by straight lines, to make them show up more clearly.

Some of the work described above will probably have been done in the primary school and further suggestions for it will be found in *The teaching of mathematics in primary schools* and *Mathematics in secondary modern schools*.

When more difficult graphs are drawn, a number of details need to be considered. These concern axes, scales, labelling, the question of intermediate points and the technique of drawing. They cannot all be explained at once and at first the teacher must give detailed instructions as the work proceeds. Thus good habits can be instilled, while difficulties are postponed. Occasions for discussion will arise from time to time and the pupils will gradually acquire a fuller understanding and the ability to decide questions of detail for themselves. The following must be kept in mind from the start, at first by the teacher, later by the pupils:

Axes. The independent variable should always be measured across the page. This of course is a pure convention, comparable with writing from left to right, but teachers should insist on a rigid adherence to it. The pupils will then become accustomed to the idea that the quantity whose variation they are exhibiting is measured up the page and, at a later stage, they will begin to appreciate the meaning of the terms 'dependent variable' and 'independent variable'.

Scales. These will at first be chosen by the teacher. It is more important to choose an easy scale than the largest possible one, and the intention to take readings in decimals should be borne in mind. Thus convenient scales are, for example, 1, 2, 5, 10 or 100 units to the inch or, for larger scales, 1 unit to 2, 5 or 10 inches. The zero of a scale should be shown where reasonably practicable. (It may be noted that gross misrepresentation can easily be achieved, as in some advertisements, by the omission of this. On the other hand it is clearly impossible to show zero for temperatures and dates, and undesirable in some other cases where small variations are being studied.) Numbers should be written along the axes, at intervals of an inch or half an inch. Where one inch represents 5 units, or 0·5 of a unit, it is a great help to put marks to show where the sub-divisions come (Fig. 10.1).

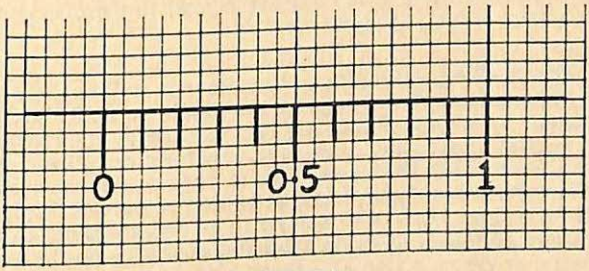


FIG. 10.1

The reading of scales is difficult, as anyone who has used a slide rule knows, and requires careful teaching.

Labelling. Every graph should have a heading, referring primarily to the dependent variable, e.g. 'speed of a train,' 'number of boys in the school from 1939 to 1963,' 'time of sunset through the year.' The axes should be labelled as, for example, 'Time in seconds', 'Number of boys', 'Speed in m.p.h.' This, together with numbers at suitable intervals, is sufficient: it is not necessary to add '1 inch represents 5 m.p.h.'

Where two or more graphs are drawn with the same axes they should be drawn, if possible, in different colours, and should in any case be suitably labelled.

Intermediate points. Examples will be met in which:

(*a*) intermediate points have no meaning;
(*b*) the intermediate points have a meaning, but their position cannot reasonably be guessed;
(*c*) the intermediate points have a meaning and the plotted points show a sufficiently regular distribution to encourage interpolation.

Graphs of type (*a*) are functions of a discrete variable, and include the times of high tide, of sunset and of lighting-up time on successive days, the size of the exterior and interior angles of regular polygons, the result of a series of experiments in which each pupil counts the number of heads in ten throws of a coin or the number of vowels in a line of print (a frequency diagram being drawn to show how many such experiments gave each possible result). In these examples the plotted points will be joined by straight lines to assist the eye in following the variation.

For (*b*), good examples are the number of people in a bus queue, counted at intervals of two minutes; the speed of a motor-car observed every half-minute; the water-level in a tidal river observed at the same time each day. Here again, the points will usually be joined, if at all, by straight lines.

Graphs of type (*c*) should include some where the variation is irregular, but interpolation is reasonably possible, e.g. growth curves, population curves, travel graphs (with uniform velocity assumed, as in the construction of railway time-tables); and some in which the variation follows an algebraical law, e.g. ready reckoners, fahrenheit and centigrade temperatures, squares of numbers, logarithms, simple and compound interest, areas of circles, length of a degree of longitude, distance of the horizon. In these graphs the points will usually be joined by a smooth curve; but, where the points lie approximately on a straight line, or where there is reason to suppose that they ought to do so, it may be more useful to draw a straight line passing as near to them as possible. For this purpose a piece of cotton may be used.

Drawing. The labelling and marking of scales may well be done in ink, but the plotting of points and the joining of them should normally be done with a sharp pencil, HB for choice. Points are best marked by the crossing of two lines, and it is often convenient, more especially for a point inside one of the squares of the graph paper, for the two lines to be parallel to the axes.

The drawing of a smooth curve through plotted points is difficult and much practice is needed. The golden rule is never to go over the same ground twice. To make it easier to follow this rule, the hand should be on the inside of the curve, the wrist or elbow acting as a pivot, and the curve may be drawn in sections, contact being broken off the moment the position becomes uncomfortable or the direction seems to be going wrong.

As the examples mentioned will have indicated, opportunities for drawing graphs occur in many parts of the arithmetic course as well as in other branches of mathematics. In algebra, graphs are of fundamental importance as a means of developing the idea of a function. Gradients, too, can be observed at a very early stage, since quite young pupils appreciate that the steepness of a curve is an indication of rate of change.

Graphs are also drawn in such other subjects as physics, biology and geography and thus provide a connecting link with mathematics and an opportunity for co-operation between teachers in different departments.

It is necessary to add that the drawing of a graph is not an end in itself. Questions should always be asked with a view to bringing out the meaning and purpose of the graph. How does such-and-such a quantity vary? Is the variation regular or irregular? If irregular, can we account for any of the noticeable features? If regular, does it follow an algebraical law?

Interpolation is not the primary purpose of every graph and should not be overworked; but it provides suitable questions in some cases. Travel graphs lead to interesting questions about trains passing each other, and an exponential graph can be used to suggest meanings for fractional indices. But, when all is said and done, a graph is primarily a method of presentation of facts which are already known.

INSTITUTE OF EDUCATION FOR WOMEN
Dept. of Extension
SERVICE
CALCUTTA-27

CHAPTER 11

SQUARES AND SQUARE ROOTS

BEFORE any work is attempted on square root the idea of a square and its associated terminology must be thoroughly understood, e.g. $7 \times 7 = 49$, or the square of $7 = 49$, or $7^2 = 49$.

The study of integers and their squares in the sequences

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & \dots \\ 1 & 4 & 9 & 16 & 25 & \dots \end{array}$$

is useful, and the squares of the early integers should be learnt by heart, possibly as far as 20^2.

The step from one square to the next may be illustrated by diagrams such as

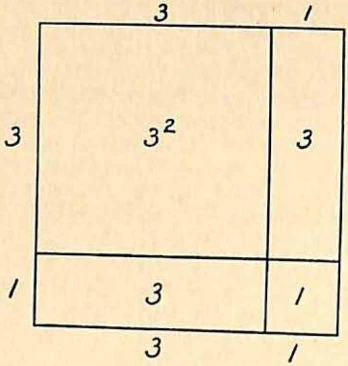


FIG. 11.1

and may be generalised to

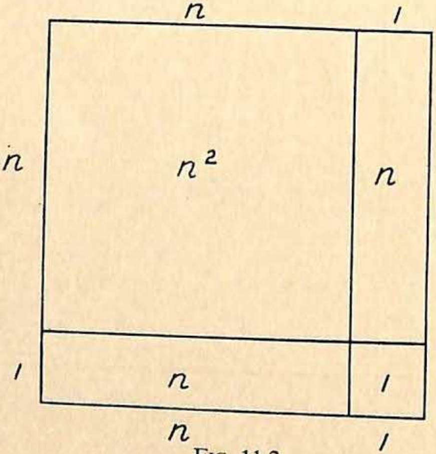


FIG. 11.2

showing the step from n^2 to $(n+1)^2$ by adding n and n and 1, for classes able to take it.

Square roots should be introduced in a situation which shows the need for them. For example, after finding the areas of squares of side 3, 5, 8, ... units it is natural to ask the converse question, 'What is the length of the side when the area is 9, 25, 64, ... sq. units?' This soon raises the problem of finding the side when the area of the square is, say, 7 sq. units. Here is a real challenge and the pupils should first be allowed to make their own crude attempts at meeting it. The table of numbers and squares shows that the side lies between 2 and 3.

Try 2·5; $2{\cdot}5^2 = 6{\cdot}25$—too small.
$2{\cdot}6^2 = 6{\cdot}76$—still too small.
$2{\cdot}7^2 = 7{\cdot}29$—too big.

So $\sqrt{7}$ lies between 2·6 and 2·7, and trial can be continued with 2·65, etc.

The graph of squares should be plotted and used for the estimation of the square roots of numbers, both integral and non-integral, which are not necessarily perfect squares. Applied to the problem of finding $\sqrt{7}$ it might give 2·65. It is easy to see that to find a better approximation, we want a drawing

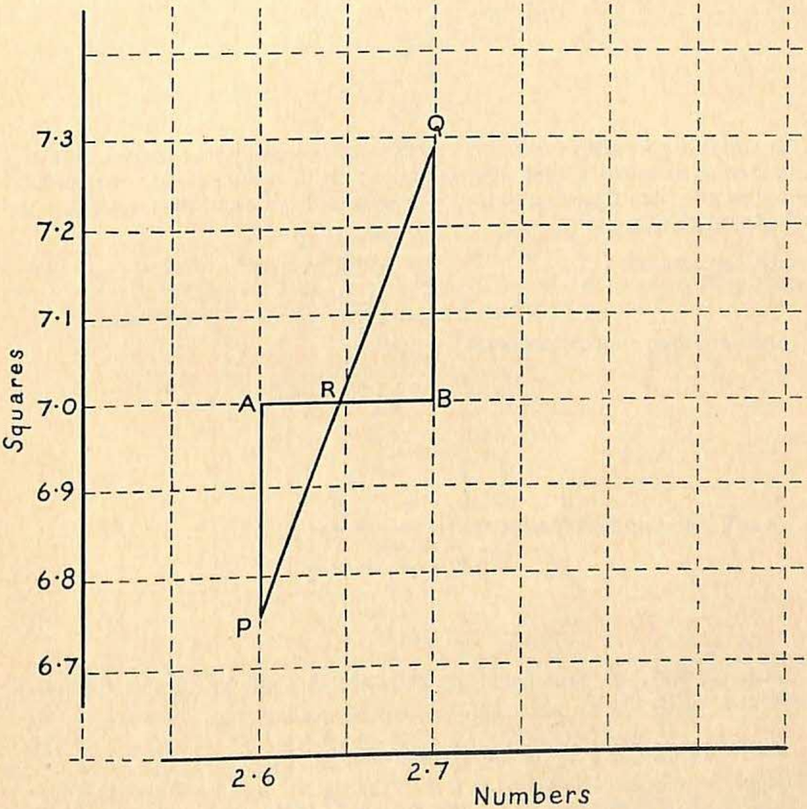


FIG. 11.3

of the graph in this neighbourhood on a larger scale. We have already seen that $2{\cdot}6^2$ is too small by 24 hundredths and $2{\cdot}7^2$ is too large by 29 hundredths. In Fig. 11.3 *AB* is drawn to represent the level of 7 on the squares axis and its length is made 10 units of the graph paper. Let *A* and *B* be above 2·6 and 2·7 on the number axis. *AP*, *BQ* are drawn to represent 24 and 29 on any convenient scale. The straight line joining *P* and *Q* represents the graph closely in this neighbourhood. It meets *AB* at *R* which is estimated to be above 2·64(6) on the number axis. This is a second approximation to $\sqrt{7}$.

The method of finding a square root by factors should not be neglected. It is useful when the number whose square root is to be found is a perfect square and also when there is a single unpaired factor whose square root is known, such as $\sqrt{2}$ or $\sqrt{3}$;

$$\begin{aligned}
\text{e.g. (i) } \sqrt{7056} &= \sqrt{(7\times 1008)},\\
&= \sqrt{(7\times 7\times 144)},\\
&= \sqrt{\{(7\times 12)\times(7\times 12)\}},\\
&= 7\times 12,\\
&= 84.\\
\text{(ii) } \sqrt{98} &= \sqrt{(2\times 7\times 7)},\\
&= 7\sqrt{2},\\
&= 7\times 1{\cdot}414,\\
&= 9{\cdot}898.
\end{aligned}$$

A simple approach to Newton's method of estimating the square root of a number is to guess a first approximation to its value, divide the guessed number into the original, and then average the divisor and the quotient to get a better approximation;

e.g. suppose $\sqrt{2}=1{\cdot}5$,

$$\text{then a better approximation is } \frac{1{\cdot}5+\dfrac{2}{1{\cdot}5}}{2},$$

$$=\frac{1{\cdot}5+1{\cdot}33\ldots}{2},$$

$$\simeq 1{\cdot}42.$$

$$\text{A still better approximation is } \frac{1{\cdot}42+\dfrac{2}{1{\cdot}42}}{2},$$

$$=\frac{1{\cdot}42+1{\cdot}4084\ldots}{2},$$

$$\simeq 1{\cdot}4142.$$

Alternatively, we can show the extraordinary power of the method by working as follows:

(*a*) work the first stage in fractions, obtaining $\frac{17}{12}$;

(*b*) then the second stage becomes $\quad \frac{1}{2}\left(\frac{17}{12}+\frac{24}{17}\right)$, and working to six places of decimals, say,

$$\begin{aligned}\text{we have } &\tfrac{1}{2}(1{\cdot}416\,666\ \ldots + 1{\cdot}411\,764\ \ldots),\\ &= \tfrac{1}{2}(2{\cdot}828\,430\ \ldots),\\ &= 1{\cdot}414\,215.\end{aligned}$$

This may now be compared with the value of $\sqrt{2}$, correct to seven significant figures, 1·414 214.

The pupils may investigate the accuracy of the approximation at any stage by squaring or by carrying the process a stage further.

This is a method for which desk calculating machines are particularly suited.

Fig. 11.2 above suggests yet another method. The difference between $(n+1)^2$ and n^2 is a little more than twice n; so, for example, $40^2 = 1600$ and $41^2 = 1681$; the difference is 81, a little more than twice 40, and $81 \div (2 \times 40)$ gives the needed 1. See Fig. 11.4:

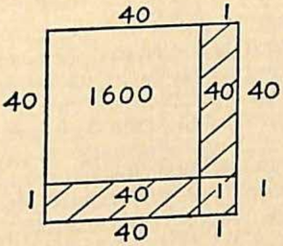


FIG. 11.4

Similarly, suppose $\sqrt{529}$ is required; this is clearly between 20 and 30. The second digit can be obtained by taking $529 - 20^2$, viz. 129, and dividing this by twice 20; $129 \div 40$ gives 3 as the start for the quotient, and 3 is the next figure of the square root; in fact $\sqrt{529} = 23$ exactly. See Fig. 11.5:

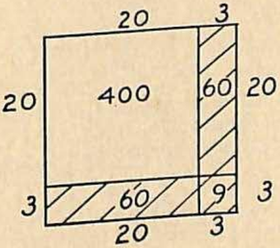


FIG. 11.5

Again, to find the square root of 1984, which lies between 40 and 50, take 40^2 from 1984, leaving 384; divide 384 by twice 40, giving 4 and a remainder, showing that 1984 lies between 44^2 and 45^2; take 44^2 (=1936) from 1984, and divide the remainder, 48, by twice 44, giving 0·5; then $\sqrt{1984} \simeq 44{\cdot}5$, and so on to any desired degree of accuracy.

It should be shown that the square of any one-digit number lies between 1 and 100, i.e. it has 1 or 2 digits; the square of any two-digit number lies between 100 and 10 000, i.e. it has 3 or 4 digits. So, reversing the process, we can show that if a number has one or two digits, its square root has one; if it has three or four, its square root has two, and so on. In general, then, if we mark off the digits in pairs from the decimal point, corresponding to each compartment there will be one digit in the square root.

E.g. of 4 | 43 | ·94 | 49 the square root is 2 | 1 | ·0 | 7
and of 4 | ·43 | 94 | 49 the square root is 2 | ·1 | 0 | 7

It should be noted also that one sequence of figures in the square will provide two different sequences of figures in the square root according to the position of the decimal point; thus 1000, 100·0, 10·00, 1·000 give as square roots 31·62, 10, 3·162, 1. This should be carefully explained and thoroughly understood when first using tables.

These facts will explain the first step in the general method. It will enable certain square roots (e.g. $\sqrt{0{\cdot}01}$, $\sqrt{0{\cdot}0064}$, $\sqrt{1{\cdot}44}$) to be written down, and is also useful in squaring mentally, e.g. 0·1 or 0·07, using the original argument that the number of decimal places in the square will be twice the number in the square root.

Considering the widespread use of tables and mechanical aids to computations nowadays, it may well be argued that there is no need for this somewhat cumbersome and lengthy process to be undertaken at all; but the actual achievement of the desired result by one's own effort undoubtedly gives real enjoyment to some and satisfies the curiosity of many, though they may never do it again. Moreover, it is an elementary example of the increasingly important method of iteration, although it is not actually one used by computers.

The general method should not be attempted before the pupils have enough algebra to understand the theory. It is suggested that an arrangement of the work might be as follows, and the first examples should come out exactly.

To find the square root of 1011·24:

	3	1	· 8	
3	10	11	·24	
	9			
61	1	11		.... (i)
		61		
628		50	24	................................(ii)
		50	24	

It is worthwhile to point out that at line (i) in the above work, the 11 124 is 101 124 − 300^2, that at line (ii) the 5024 is 101 124 − 310^2, and so on.

To find the square root of 1984, correct to 2 decimal places:

	4	4	· 5	4	(2)
4	19	84	·00	00	00
	16				
84	3	84			
	3	36			
885		48	00		
		44	25		
8904		3	75	00	
		3	56	16	
89082			18	84	00
			17	81	64

∴ $\sqrt{1984} = 44{\cdot}54$.

The process of dividing by twice the part of the root already obtained has been suggested by the diagrams given above. We may also illustrate the process by showing $(a+b)^2$ divided into 4 parts and using $(a+b)^2-a^2=2ab+b^2$, while $\frac{2ab+b^2}{2a}=b+$(a fraction), if b is sufficiently small.

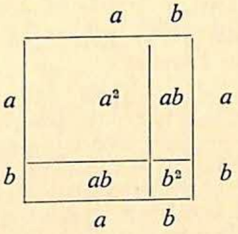


FIG. 11.6

Special attention should be given to finding square roots of decimals with one significant figure, e.g. $\sqrt{0{\cdot}1}$, $\sqrt{0{\cdot}6}$, $\sqrt{0{\cdot}06}$, $\sqrt{0{\cdot}0005}$.

When fractions are considered several methods may be employed. For example:

$\sqrt{1\frac{9}{16}}=\sqrt{\frac{25}{16}}=\frac{5}{4}=1\frac{1}{4}$;

$\sqrt{1\frac{1}{2}}=\sqrt{1{\cdot}5}=1{\cdot}225$ ... by tables or a suitable method of approximation;

$\sqrt{1\frac{1}{2}}=\sqrt{\frac{3}{2}}=\frac{1}{2}\sqrt{6}$, etc.

Their relative advantages can be discussed and pupils should be encouraged to make their own choice.

Full understanding of the notation often takes a long time. This can be seen, for example, in such clumsiness as evaluating $(\sqrt{2})^2$ by squaring $1{\cdot}414$. It is particularly important throughout work in square roots to test the results frequently by squaring. This can help to make clear such relations as

$$\sqrt{0{\cdot}04}=0{\cdot}2, \quad \sqrt{15}=\sqrt{3}\times\sqrt{5}, \quad \sqrt{12}=2\sqrt{3}, \quad \sqrt{10}>3.$$

Squaring is also useful in correcting errors, e.g.

$$\sqrt{(9+4)}\neq 3+2 \quad \text{because} \quad 9+4\neq 25.$$

Applications of Pythagoras' theorem to plane and solid figures make useful exercises, and it may come as a surprise to a pupil to find that a line $\sqrt{2}$ inches long can be drawn on paper but cannot be measured exactly with a ruler. It is useful to emphasise that the statement '$\sqrt{2}=1{\cdot}414$ ...' is a way of saying that '$\sqrt{2}$ lies between $1{\cdot}4135$ and $1{\cdot}4145$.' The idea of 'betweenness' deserves more attention than it commonly receives (see Chapter 13), and it may well be mentioned that $\sqrt{2}$ is a new kind of number, an irrational one, possibly the first such number that the pupil has met.

Many who teach Pythagoras' theorem early would introduce this work on square roots at about the same time in order to solve problems arising from the theorem. It is also needed in the solution of quadratic equations, and in the application of certain formulae. An example is the formula $\sqrt{\frac{3h}{2}}$ for the distance in miles of the visible horizon from a height h feet above sea-level, which leads to such questions as to whether Ireland can be seen from the top of Snowdon or Scafell.

Pupils who have learned to factorise the difference of two squares in algebra may be interested if they are shown how a continued fraction can be used to estimate a square root,

e.g.

$$\begin{aligned}
\text{if } x^2 &= 2,\\
x^2-1 &= 1,\\
(x-1)(x+1) &= 1,\\
\text{so}\qquad x-1 &= \frac{1}{1+x}\\
\text{and}\qquad x &= 1+\frac{1}{1+x} = 1+\cfrac{1}{1+1+\cfrac{1}{1+x}}\\
&= 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}}}
\end{aligned}$$

The practical application of this idea consists in making an approximation by terminating the continued fraction at some stage. The pupil may be led to discover experimentally that the more terms that are retained, the better the approximation. Denoting successive approximations by $a_1, a_2, a_3, \dots$ we have

$$a_1 = 1+\tfrac{1}{2} = \qquad 1{\cdot}5.$$

$$a_2 = 1+\frac{1}{2+\frac{1}{2}} = 1+\frac{2}{5} = \qquad 1{\cdot}4.$$

$$a_3 = 1+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}} = 1+\cfrac{1}{2+\frac{2}{5}} = 1+\frac{5}{12} = 1{\cdot}42 \text{ approx.}$$

$$a_4 = 1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}}} = 1+\cfrac{1}{2+\frac{5}{12}} = 1+\frac{12}{29} = 1{\cdot}414 \text{ approx.}$$

Tables. To return to the argument that the freer use of logarithm and square root and other tables makes any attention to the long methods here described unnecessary: individual teachers must decide this in terms of their own scale of values and the circumstances in which they work. Ultimately one hopes that pupils will have as near a universal view of mathematics as possible, and that anything they learn will be intrinsically worth knowing. Even if square root tables are accepted and used, the question of how the tables are compiled might reasonably be asked, and pupils could usefully construct a line or two. They may be shown both forms of the tables, and reminded that a table of squares used in reverse will provide square roots. And even when they calculate square roots using logarithms, they may be informed that Henry Briggs calculated the first set of logarithm tables in England by truly heroic feats of the square root process. In any case, ideally, a pupil should have at his disposal as many tools as he can reasonably master at any stage of his work.

CHAPTER 12

LOGARITHMS

PROFESSOR JOHN PERRY in his 'lectures to working men' said, 'Some friends of mine assert that no man or boy ought to be allowed to use logarithms until he knows how to calculate them. . . . This is a hard saying. It is exactly like saying that a boy must not wear a watch or a pair of trousers until he is able to make a watch or a pair of trousers.' Many teachers today would go further and say that it is quite practicable to learn to use a table of logarithms as a tool of calculation before understanding how the process works or how it came to be invented. In this approach the slide rule makes a convenient starting point especially if home-made models are constructed from strips of logarithmic graph paper. The aim should be to acquire through practical experience some understanding of the law which is formulated later in the form $\log ab = \log a + \log b$. The use of the tables is then easily developed.

Many children in grammar schools, however, will derive greater satisfaction from an introduction through indices, because this gives some understanding of underlying principles from the outset. It is not necessary to have much previous knowledge of negative and fractional indices. Negative integral indices can be introduced by inspecting the table

$$\begin{array}{cccc} \ldots & 10^3 & 10^2 & 10^1 \\ \ldots & 1000 & 100 & 10 \end{array}$$

and noting that dividing by 10 in the second line is equivalent to subtracting 1 from the corresponding index in the top line. A continuation of this process leads to

$$\begin{array}{ccc} 10^0 & 10^{-1} & 10^{-2} \ldots \\ 1 & 0{\cdot}1 & 0{\cdot}01 \ldots \end{array}$$

Some practice with positive and negative indices can be given by expressing large and small numbers in standard form and performing simple calculations with them. Standard form will also be useful later on in determining characteristics.

The approach through indices. We can begin with the bald statement that at the beginning of the seventeenth century a Scotsman called John Napier devised a method of multiplying two numbers by adding two other numbers, a much simpler operation. We can then ask if examples of such a process have ever come within our pupils' experience and eventually obtain examples such as $2^3 \times 2^2 = 2^5$. Then a table of powers of 2 from, say, $\frac{1}{8} = 2^{-3}$ to $4096 = 2^{12}$ can be written on the board and used to multiply any two numbers in the table, provided that their product also falls within its range. The problem of how to apply the method to other numbers will soon be raised and can be solved, as a first approximation, by drawing the graph of powers of 2.

In effect this graph introduces such approximations as $7 = 2^{2{\cdot}8}$ and is a temporary substitute for a definition of fractional indices. In the unlikely event

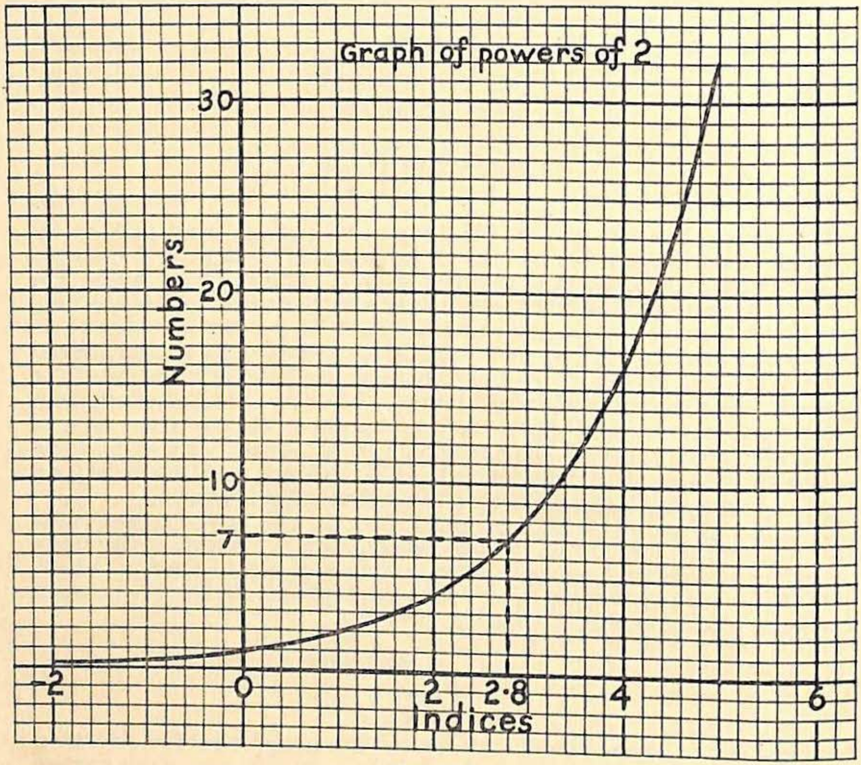


FIG. 12.1

of a pupil asking how 2·8 two's can be multiplied we must explain the need of a fresh definition and examine a few particular cases, e.g.

$$2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2, \text{ and so we define } 2^{\frac{1}{2}} \text{ to be } \sqrt{2},$$

$$2^{\frac{2}{3}} \times 2^{\frac{2}{3}} \times 2^{\frac{2}{3}} = 2^2, \text{ and so we define } 2^{\frac{2}{3}} \text{ to be } \sqrt[3]{2^2}.$$

General definitions and manipulative exercises with fractional indices are best postponed, since they distract attention from the main purpose.

Dissatisfaction with the accuracy and range of the graph will soon be expressed, and then the logarithm table can be introduced. We must first point out that powers of 10 are more convenient than powers of 2 and the story of Briggs' journey to Edinburgh to discuss this point with Napier is worth telling here.*

We can demonstrate the advantage of the change of base as soon as the tables can be read. From the graph, $7 = 2^{2\cdot8}$ and $70 = 2^{6\cdot1}$ approximately; from the tables, $7 = 10^{0\cdot85}$ approximately. Hence $70 = 10^1 \times 10^{0\cdot85} = 10^{1\cdot85}$. If the base 10 is used, the decimal part of the logarithm is not altered by a change in the position of the decimal point of the number, and this greatly reduces the bulkiness of the tables.

Use of the table is conveniently introduced in three stages: (i) for numbers between 1 and 10, (ii) for numbers greater than 10, (iii) for positive numbers less than 1.

---

* F. Cajori, *A history of mathematics.* (Macmillan), p. 150.

Stage (*i*). Some teachers like to use an index method of setting out for some time, others go over to a tabular setting out immediately or after only one or two lessons. The former school believes that reiteration of index statements leads in the end to a better understanding of the nature of logarithms; the latter claims that logarithms should be regarded as a tool and the first consideration is the most efficient use of this tool. Even if the index method is used, pupils should be trained from the start to add the logarithms by writing one underneath the other in the margin.

It will be found useful in preparation for Stage (ii) to allow some answers to exceed 10. For example,

$$\begin{array}{rll} 6{\cdot}123 \times 3{\cdot}947 & = 10^{0\cdot7870} \times 10^{0\cdot5963}, & \quad 0{\cdot}7870 \\ & = 10^{1\cdot3833}, & \quad \underline{0{\cdot}5963} \\ & = 10^{1} \times 10^{0\cdot3833}, & \\ & = 10 \times 2{\cdot}42, & \quad 1{\cdot}3833 \\ & = 24{\cdot}2. & \end{array}$$

Stage (*ii*). A few examples like $2{\cdot}8 = 10^{0\cdot4472}$, $28 = 10 \times 2{\cdot}8 = 10^{1\cdot4472}$, $280 = 10^2 \times 2{\cdot}8 = 10^{2\cdot4472}$ etc. together with an examination of the table $10 = 10^1$, $100 = 10^2$, $1000 = 10^3$, quickly lead to the following table of characteristics:

Range of numbers	1–10	10–100	100–1000
Characteristic	0	1	2

This approach may usefully be supported by sketching the graph of powers of ten on the blackboard. This rubs in the interpolation based on the above table and shows in the most striking way the relationship between numbers and their logarithms. Difficulties of scale and the limited number of points which it is possible to plot (without begging the question by using the table) make this an unsuitable exercise for the pupils unaided.

It may also be pointed out that the characteristic is easily found if the number is expressed in standard form. For example if 23 800 is written in the form $2{\cdot}38 \times 10^4$, it is immediately clear that the characteristic is 4.

Stage (*iii*). This can be introduced (not necessarily immediately after Stage (ii)) by going from $2{\cdot}8 = 10^{0\cdot4472}$ to $0{\cdot}28 = 10^{-1} \times 2{\cdot}8 = 10^{-1+0\cdot4472}$. The pupil expects this index to be written as an ordinary negative number $-0{\cdot}5528$ and the disadvantage of doing this can be brought home forcibly by setting pairs of exercises such as: Find the number whose logarithm is (i) $-0{\cdot}5528$, (ii) $-1+0{\cdot}4472$. As soon as the need to write numbers in this unusual form is appreciated some teachers introduce the bar notation. Others pass through a transition stage in which the characteristic and mantissa are written separately. For example,

$$\begin{aligned} 0{\cdot}0048 \times 0{\cdot}58 &= 10^{-3} \times 4{\cdot}8 \times 10^{-1} \times 5{\cdot}8 \\ &= 0{\cdot}002\,784 \text{ approx.} \end{aligned}$$

$$\begin{array}{l|l} 0{\cdot}0048 & -3 + \ \cdot6812 \\ 0{\cdot}58 & \underline{-1 + \ \cdot7634} \\ & -4 + 1{\cdot}4446 \\ 0{\cdot}002784 & -3 + \ \cdot4446 \end{array}$$

This separation is particularly useful when a negative characteristic has to be subtracted, a case that often proves troublesome at first.

$$\frac{0{\cdot}875}{0{\cdot}957}=0{\cdot}914 \text{ to 3 sig. fig.}$$

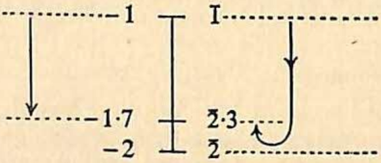
$$\begin{array}{l|l} 0{\cdot}875 & -1+{\cdot}9420\\ 0{\cdot}957 & -1+{\cdot}9809\\ \hline 0{\cdot}9143 & -1+{\cdot}9611\end{array}$$

The working is different according to the method of subtraction used, e.g.

$$\begin{array}{r} -2+1{\cdot}9420\\ -1+\ {\cdot}9809\\ \hline -1+\ {\cdot}9611\end{array} \qquad\text{or}\qquad \begin{array}{r} -1+1{\cdot}9420\\ 0+\ {\cdot}9809\\ \hline -1+\ {\cdot}9611\end{array}$$

(It may be pointed out that this calculation is performed more easily by writing the fraction in the equivalent form $\frac{8{\cdot}75}{9{\cdot}57}$.)

The bar notation should be seen as a convenient but not indispensable part of logarithmic calculation. At first some practice with equivalences like $-1{\cdot}7=\bar{2}{\cdot}3$ is desirable and a diagram may be useful.



Much practice with exercises such as $\bar{2}+\bar{3}$, $\bar{3}-\bar{2}$, $\bar{2}-\bar{3}$, $\bar{3}{\cdot}3-\bar{3}{\cdot}2$, $\bar{3}{\cdot}2-\bar{2}{\cdot}3$, etc. will be necessary. It must be remembered that two or more methods of subtraction may be in use in the class. For instance $\bar{2}{\cdot}5-\bar{3}{\cdot}7=0{\cdot}8$ may be worked by saying '7 from 15 is 8, $\bar{3}$ from $\bar{3}$ is 0;' by saying '7 from 15 is 8, $\bar{2}$ from $\bar{2}$ is 0;' or by saying '8 must be added to 7 to make 15, add 1 to $\bar{3}$—this makes $\bar{2}$, 0 must be added to $\bar{2}$ to make $\bar{2}$.'

The theory of logarithms. This hardly comes within the scope of arithmetic at all, and yet important properties of the logarithm function are self-evident from its graph and an informal discussion of them is probably useful. In any case the ability to draw a quick sketch of $y=\log_{10} x$ is valuable. Some of the properties which may be mentioned at this stage are:

(i) The logarithm of a number greater than 1 is positive; the logarithm of a number between 0 and 1 is negative; the logarithm of a negative number is not defined. $\log 1=0$.

(ii) The logarithm increases steadily as the number increases and therefore no two numbers have the same logarithm. Also every index is the logarithm of one and only one number.

(iii) It follows from (ii) that the logarithm of any number between, say, 10^2 and 10^3 is $2+h$, and of any number between, say, 10^{-2} and 10^{-3}, $-3+k$, where h and k are positive fractions. Hence the graph summarises the process of determining the characteristic. If it is thought necessary to give a rule for finding the characteristic, the simplest is probably one based on the expression of the number in standard form. The characteristic is the displacement from the *units* place of the first (i.e. the left hand) significant figure reckoned positively if it is to the left of the units place, negatively if it is to the right.

Powers and roots of a positive number smaller than 1. When the characteristic is negative, the process of finding a root is sometimes a little troublesome. It may be worth pointing out to teachers that it is convenient to choose for the first calculation of a root the converse of a previous calculation of a power:

(i) *Evaluate* $(0\cdot0555)^2$.

0·0555	$\bar{2}\cdot7443$
	2
0·00308	$\bar{3}\cdot4886$

Details of working

$(-2+\cdot7443)\times2$
$=-4+1\cdot4886$
$=\bar{3}\cdot4886$

(ii) *Evaluate* $\sqrt{(0\cdot00308)}$. We retrace the previous working step by step in the opposite order.

0·00308	$\bar{3}\cdot4886$
0·0555	$\bar{2}\cdot7443$

Details of working

$\bar{3}\cdot4886=-4+1\cdot4886$
Half of this is $-2+\cdot7443$.

Antilogarithms. There is a division of opinion on the use of antilogarithm tables. Some teachers hold that, as there are no 'anti-trigonometrical tables', one has to learn to use most tables both ways, and it is best to do this with the logarithm table as well. To encourage the use of two tables is only to introduce another possible source of error.

Against this it is argued that it is important to acquire the habit of glancing at the heading of a table before using it. (Some of us only learn this when we have tried to catch a Sunday train on a weekday.) Numerical tables are designed to be read forwards and they are always more troublesome to read backwards. The antilogarithm table saves some of this trouble and is quicker. Teachers who hold this view therefore encourage its use from the beginning.

The setting out of logarithmic calculations. Since the purpose of logarithms is mainly to ease the burden of calculation, the setting out should be as simple and economical as is consistent with order and accuracy. The fundamental principle is that addition or subtraction is likely to be more accurate if it is done 'up and down' rather than 'across' the page.

The logarithmic work should be separated from the statement of the calculation and should not be crammed into a margin. In particular, it is wise to leave plenty of room for developments on the logarithm side of the table. The following examples are offered as general suggestions, though many teachers would probably want to make modifications in detail.

$$\frac{18\cdot7\times15\cdot8}{216\times0\cdot047}=29\cdot1 \text{ to 3 sig. fig.}$$

18·7	1·2718	
15·8	1·1987	
Numerator	2·4705	2·4705
216	2·3345	
0·047	$\bar{2}\cdot6721$	
Denominator	1·0066	1·0066
29·10		1·4639

$$428{\cdot}5\sqrt{\frac{0{\cdot}8512}{0{\cdot}9284}}=410{\cdot}2 \text{ approx.}$$

0·8512	$\bar{1}{\cdot}9300$
0·9284	$\bar{1}{\cdot}9677$
Quotient	$\bar{1}{\cdot}9623$
Square root	$\bar{1}{\cdot}9811$
428·5	2·6319
410·2	2·6130

It is always useful to scrutinise a result for reasonableness. Here the square root is obviously a little less than 1 and the answer must be a little smaller than 428·5.

The calculation of part of the logarithm table. This can generate considerable interest and is well worth doing when some understanding of fractional indices has been reached. There are several methods:

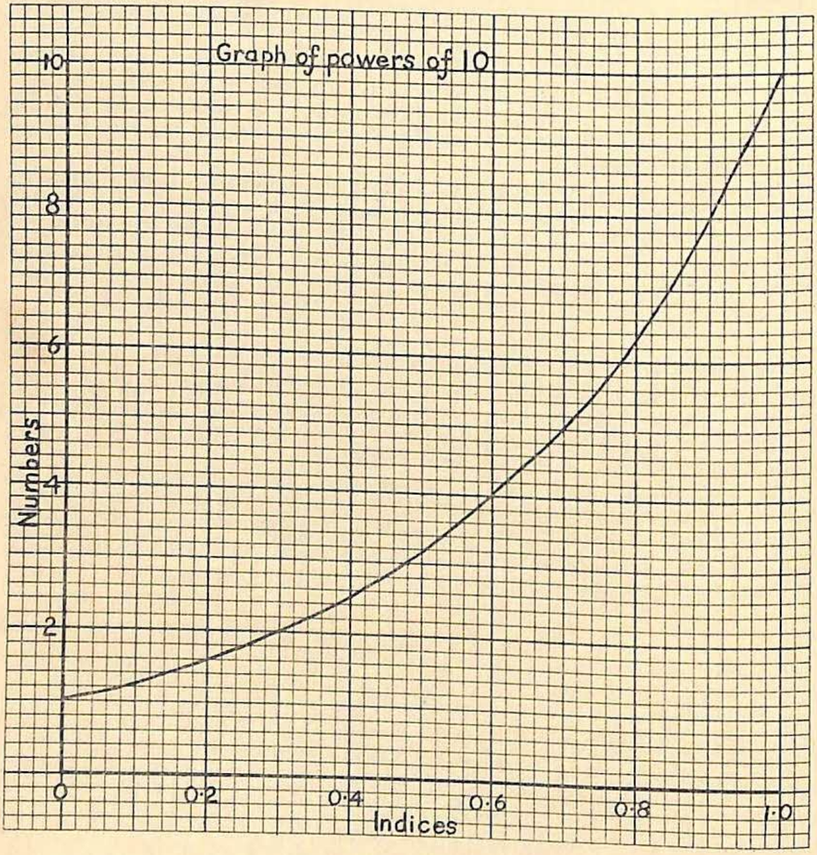


FIG. 12.2

(i) Logarithms can be deduced from powers of a number near 1, e.g. 1·1. This method is fully explained in *The teaching of algebra in schools* on pp. 50–52, and it is particularly suitable if a calculating machine is available.

(ii) Rougher approximations can be obtained from a graph of powers of 10. We illustrate by finding the logarithms of the integers from 1 to 10. By repeated use of square roots we find $10^{\frac{1}{2}} = 3{\cdot}162$, $10^{\frac{1}{4}} = 1{\cdot}779$, $10^{\frac{1}{8}} = 1{\cdot}333$. Then $10^{\frac{3}{8}} = 10^{\frac{1}{4}+\frac{1}{8}} = 10^{\frac{1}{4}} \times 10^{\frac{1}{8}} = 1{\cdot}779 \times 1{\cdot}333 \simeq 2{\cdot}372$. Approximations to $10^{\frac{5}{8}}$, $10^{\frac{3}{4}}$, $10^{\frac{7}{8}}$ are found similarly and a graph plotted from these values easily gives the logarithms of the integers from 1 to 10 correct to 2 decimal places (Fig. 12.2).

(iii) A refinement of method (ii) yields 4-figure accuracy without too much work and, of course, if this can be secured it is much more impressive and greatly increases interest. Consider first the logarithm of 2. We begin by looking for a power of 2 which can also be expressed approximately as a power of 10; a good one is $2^{10} = 1024 \simeq 10^3$. We write this as

$$2^{10} = 10^3 \times 1{\cdot}024,$$

and seek an approximation to 1·024 as a power of 10 by the process of repeated extraction of square roots. Continuing the process given in method (ii) we have

Index of 10	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
Number	1·154	1·074	1·036	1·018	1·009

In the last three pairs of this table, the indices in the top line are proportional to the decimal parts of the numbers in the bottom line. Thus within the

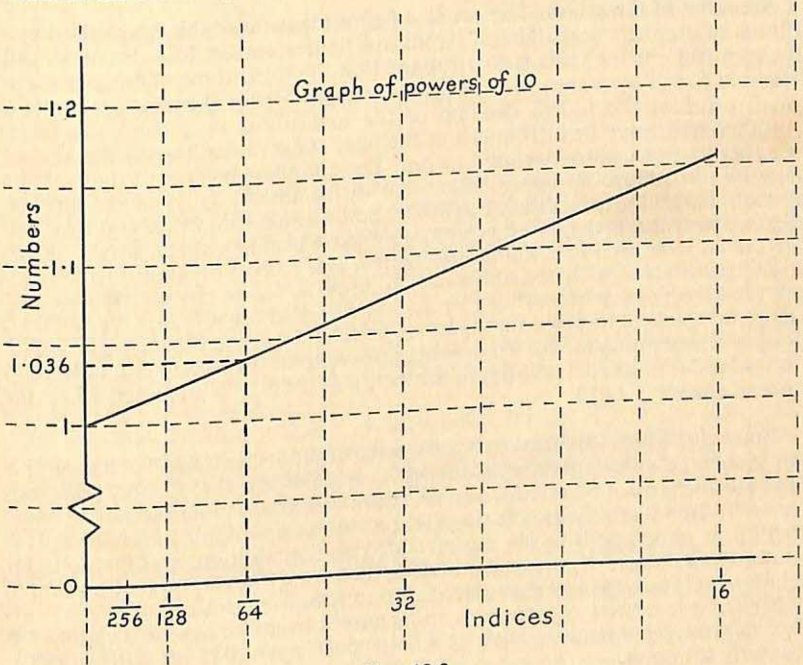


FIG. 12.3

accuracy of our calculations the part of the graph from the number 1 to the number 1·036 is a straight line (Fig. 12.3). Now 1·024 is within this range and, as it happens, 0·024 is two thirds of 0·036. Therefore, the index of 10 which gives 1·024 is $\frac{2}{3} \times \frac{1}{64} = 0\cdot0104$, to four decimals.

Therefore $2^{10} = 10^3 \times 10^{0\cdot0104} = 10^{3\cdot0104}$

Therefore $2 = 10^{0\cdot30104}$ which has an error of only 1 in the fifth decimal.

Good practice with indices can be had by calculating the logarithms of the next few primes. (It will, of course, be pointed out that the logarithms of composite numbers do not present a fresh problem.) The value 0·47711 for log 3 can be obtained from $3^4 = 81 = 1\cdot0125 \times 80 = 1\cdot0125 \times 2^3 \times 10$.

log 5 is calculated as log 10 − log 2.

The value 0·84510 for log 7 can be found from

$$7^4 = 2401 \simeq 1\cdot0004 \times 2^3 \times 3 \times 10^2.$$

It is also instructive to obtain the rougher approximation 0·84514 from $7^2 = 49 = 0\cdot98 \times 50$.

Finally it should be pointed out that these are elementary methods suitable for the calculation of just a few logarithms. If the question is asked, 'How are the actual tables made?' the short answer is that a number of comparatively widely-spaced basic values are calculated by means of logarithmic series and by using the factors of composite numbers, and intermediate values are found by means of interpolation formulae. Both these stages are facilitated nowadays by the use of calculating machines, but it is noteworthy that in 1717 Abraham Sharp published a table giving, to 61 decimal places, the logarithms of all numbers from 1 to 100 and of prime numbers from 100 to 1097.

Accuracy of the tables. The usual 4-figure logarithm table gives the logarithms of numbers containing 3 significant figures correct to 4 decimals. But for compactness the table is so arranged that the logarithms of 4-figure numbers are found by means of average differences. Hence there is sometimes a small error in the fourth decimal of the logarithms read from the table. Another error may be introduced in the final stage of reading off the answer whether antilogarithms are used or not. The cumulative effect of these errors may be to produce an appreciable error in the fourth figure of the answer, though this seldom exceeds 2. Pupils should be aware that in general this final figure is untrustworthy, but it is a pity to make a firm rule that answers should always be shortened to 3 significant figures. It is sometimes useful to have all four figures given with the abbreviation 'approx.' added as a reminder that the final figure may be out by 1 or 2. Some teachers favour writing the final figure in brackets to show that it is not as reliable as the others, e.g. 27·1(6). It is, of course, possible for the 3-figure answer to be incorrect, and, indeed, to be less accurate than the 4-figure answer, e.g. if 27·16 is obtained when the correct answer is 27·14.

Notes. (1) Some teachers advocate the use of slide rules instead of logarithms for incidental calculations up to about O level of the G.C.E. and point out that much time can be saved this way. An introduction to its use can be made in much the same way as was suggested at the beginning of the chapter. It is helpful to some pupils to use a card slide rule for addition, and then one for multiplying powers of, say, 2, calibrating their own from a graph of powers of 2 before going on to use a manufactured one. (See Fig. 12.4.)

(2) Care is always necessary in using tables to guard against copying the figures wrongly. Mistakes such as 4722 for 4772 or 2837 for 2387 are particularly common.

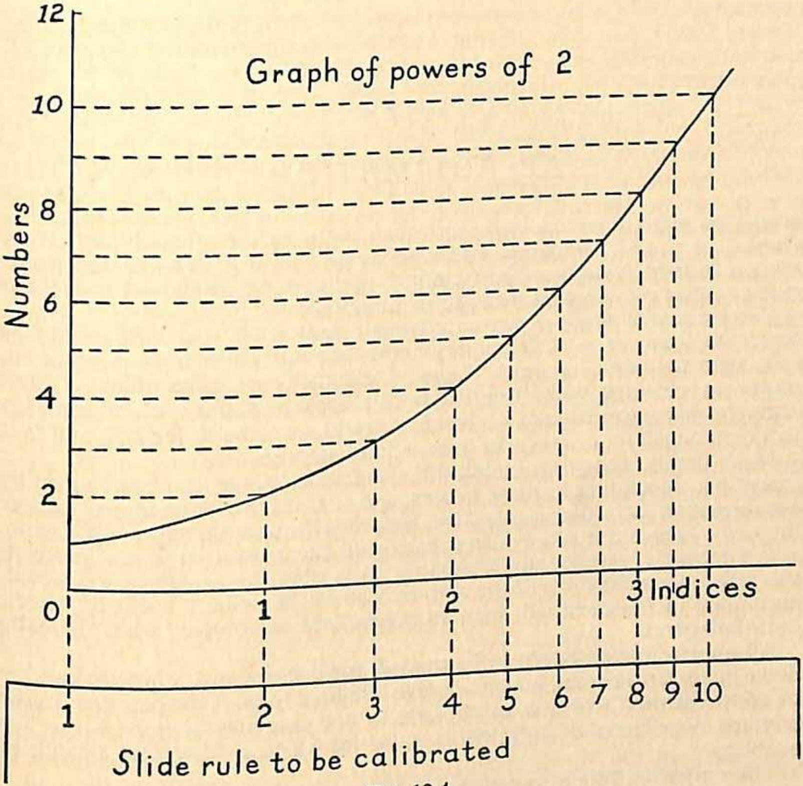


FIG. 12.4

(3) A refinement in the use of differences which can be introduced at a late stage consists in working from the nearer entry in the main table. Thus log 2188 is likely to be found more accurately as well as more quickly by subtracting 0·0004 from log 2190 rather than by adding 0·0016 to log 2180. In any case the habit of checking that the logarithm obtained after applying the difference lies between the adjacent entries in the table is an excellent one, especially in readiness for the use of tables where differences are subtracted.

(4) When looking up antilogarithms greater accuracy can be obtained by using half of the logarithm table and half of the anti-logarithm table, the half with the small differences.

(5) It should be known that 5-, 6- and 7-figure logarithm tables are available when greater accuracy is needed, as for example in navigation. Some teachers start their pupils off on 3-figure logarithms. Some experience of different tables is desirable to put the 4-figure tables in their proper perspective.

(6) A certain amount of practice in the evaluation of formulae by tables of various kinds is needed. For example, in $\frac{a^2+b^2-c^2}{2ab}$, the numerator can conveniently be calculated by tables of squares and the calculation can then be completed by logarithms.

(7) Separate tables of logarithms are much more convenient than tables at the back of a text book and should be supplied whenever possible.

CHAPTER 13

APPROXIMATIONS

IDEAS on approximation tend more than those on the other subjects to be vitiated by slovenly thinking. The superstition that if nothing is perfect one thing is as good as another is not confined to the young. Mathematics ought to make a better contribution than this to human life.

The important thing about approximate answers is that they should be correct as far as they go. If we only know that Jones lives in Hampshire, we can begin to locate him with 'England, South Coast, Hampshire, . . .' but must not continue with 'Bournemouth'; each new piece of information improves our knowledge of his residence until we come to the last, and that will be wrong if, for instance, he lives in Southampton.

Many pupils (and others) think that increased accuracy can be obtained by carrying a calculation to more figures, and that it is virtuous to give greater accuracy than that which is asked for. But when the data are only approximate, the extra figures are often wrong and spoil the answer. If Jones' room in Southampton is marked approximately by a point on a map of the world, successive magnifications of the map may show the point in a district, street, and house in Bournemouth with an appearance of accuracy which is really quite fallacious.

All approximation needs to be treated intelligently and, while we can lay down guiding principles, we cannot give infallible rules. A method or a degree of approximation which is appropriate to one case may be quite wrong for another. An element of judgment is called for which children find difficult to acquire.

The unit in which a measurement is given often indicates the accuracy involved. A distance of 56 miles between two towns is probably given to the nearest mile and the unit chosen is appropriate: it would be ridiculous to give it to the nearest yard. On the other hand the distance of the Sun, 93 000 000 miles, is most appropriately given in miles because we do not use a unit between a mile and the distance of the Sun, but there is no suggestion that the distance is given to the nearest mile.

When something is being measured it has to be decided how to record the result of the measurement. An infant-school child may well be asked to measure the length of a book to the nearest inch or the length of a room to the nearest pace, but the height of the child may be recorded in completed inches and fractions of an inch. Time, too, is measured in different ways: a person will often give the time of day to the nearest five minutes; and age is given to the nearest year or month below the age of a person.

Care should be taken when recording a reading from a calibrated instrument. A clinical thermometer can and should be read to the nearest mark, that is to the nearest 0·2° F. However, although the speedometer of a car may be calibrated to the nearest m.p.h., it is required by law only to be correct within 10% at speeds in excess of 10 m.p.h. and frequently a speedometer fails to register any speed when the vehicle is travelling at a walking speed. On the other hand the medium wave band on a wireless set is often calibrated at every ten metres and adjustment for good reception corresponds to an accuracy of about one metre. A common fault in reading a scale is to write 0·125 (3 sig.

fig. implied) when, in fact, all that can be read off is, 'About half way between 0 and $\frac{1}{4}$.' 0·1 would probably be a better answer.

In mathematics care should always be taken to state how accurately any given measurement has been made and often this will be done by using significant figures or decimal places. These two ways of expressing a degree of accuracy tend to confuse pupils, so it is better to introduce one well before the other. Decimal places are found easier and may first arise in connection with division of fractions. $\frac{1}{3}=0{\cdot}3333\ldots$, which cannot be written in full, but $\frac{1}{3}=0{\cdot}33$ correct to the nearest hundredth, or to two places of decimals, and so on.

If pupils have been used to giving an answer correct to so many places of decimals, they wonder what is the point of having a different way of expressing their answer with significant figures. There are two fundamental ideas involved here. The first is that the number of significant figures in a measurement gives an indication of the difficulty involved in making the measurement. Thus, given the appropriate apparatus, it is about equally easy to measure a length to two significant figures whether it be about 5 inches or 50 yards; and three significant figures requires a good deal more care in both cases. To measure a length correct to four significant figures will require special apparatus not generally available in the classroom.

The other fundamental idea involved in significant figures may be brought out by an example like the following. Consider an ideal circle of radius 4 inches. If we take the value of π to 1, 2, …, 6 significant figures, the following results are obtained from calculating the circumference of the circle:

Value for π	No. of sig. fig.	Circumference in inches
3	1	24
3·1	2	24·8
3·14	3	25·12
3·142	4	25·136
3·141 6	5	25·132 8
3·141 59	6	25·132 72

Now observe that, if the length of the circumference were rounded off to the same number of significant figures as the value of π, the answer would be, with one exception, correct to that order of accuracy. This illustrates the idea that significant figures roughly give the order of accuracy which can be carried through a calculation. As a general rule, a result should not be given to a greater number of significant figures than the *least* accurate of the data.* If, for instance, we calculate the area of a rectangle given as 13·6 in. by 2·7 in. the area may be stated as 36·72 sq. in., but if the data were only correct to the nearest tenth of an inch, as is implied by the way in which they are written, the area may be anywhere between

$$13{\cdot}55 \times 2{\cdot}65 \text{ sq. in.} = 35{\cdot}9075 \text{ sq. in.}$$
$$\text{and } 13{\cdot}65 \times 2{\cdot}75 \text{ sq. in.} = 37{\cdot}5375 \text{ sq. in.}$$

In this case, where we might expect two-figure accuracy, the second figure is uncertain; here we may write the answer, 36·72 sq. in., as, 37 sq. in. approx.

Significant figures are best introduced with integers rather than decimals to avoid confusion with decimal places. The distance of the moon is a good starting point: a member of the class may be able to give 240 000 miles (two significant figures) and the fact that the distance varies will probably arise in discussion. The average distance between the centres of the Earth and Moon has been given as 238 857 miles (six significant figures), the distance varying from 221 463 miles to 252 710 miles. The question about when a zero is significant will soon crop up and examples from various sources can help to clarify the pupils' ideas: the road distance from Aberdeen to Blackpool is 308 miles, to Truro is 650 miles, and to Swansea is 500 miles (all zeros significant, the distances being measured between selected points in the towns); the distance travelled by light in one second is 186 000 miles (all zeros not significant: 186 271 miles is a better figure); the mean distance of the sun from the earth is 93 004 000 miles (two zeros significant, three not). At a later stage, the idea that zeros are significant only when they have to be retained for any position of the decimal point can be brought out by writing 6·05 m as 0·006 05 km and 6050 mm.

---

* An exception to this is in finding averages.

Pupils should be encouraged to think for themselves how accurate their answers are. Cases will arise in which the best answer that a pupil can get is not expressible in any regularly used form such as 'to the nearest penny', 'to 3 significant figures', or 'to the nearest minute'. If, for instance, half the answer has been found to be £2 4s. 8d. to the nearest penny, the answer may be given as £4 9s. 4d. within a penny. Or if the angles A and B of a triangle are 43° 13′ and 77° 20′ (to the nearest minute), then $C = 59° 27'$ within a minute, but not necessarily to the nearest minute. This should not be the regular form for answers but should be accepted when circumstances prevent the use of the ordinary forms.

The use of tables* often gives pupils a false sense of accuracy. Calculating the hypotenuse of a triangle with two sides 2·8 in. and 4·1 in., the answer 4·965 in. is obtained but it is unreasonable to give more than two significant figures in the answer when the data are known to be given to 2 sig. fig.

Many mistakes in the use of tables can be avoided by having some idea of the magnitude of the value being found, particularly when there is a possibility of having turned to the wrong set of tables by mistake, as with logarithms and anti-logarithms. The decimal point, too, in looking up the square of a number, is best found by an approximation, e.g. $0{\cdot}087^2$ is a little less than $0{\cdot}09^2 = 0{\cdot}0081$. From tables, $0{\cdot}087^2 = 0{\cdot}007\,569$. As with square roots, the approximation is best done *before* consulting the tables.

In everyday life, calculating how much of some material to order, whether it be cloth, wood, paint, lino, fertiliser etc., the precise amount needed for the job seldom matters because in practice it is wise to have a certain amount to spare, and only occasionally are the materials cut to exact sizes for customers. This aspect of calculations tends to be overlooked in many text-books. The need for people to be skilled at precise calculations has been reduced by the increasing use of graduated weighing machines, ready reckoners and calculating machines. But it is still important that all calculations should be accurate to whatever degree of approximation is being used.

One of the easiest and most widespread ways of making an approximate calculation is to use a slide rule. Time is saved this way and the pupils are forced to make rough approximations (see p. 88) in order to place the decimal point. Their use raises an interesting topic for discussion, namely the degree of accuracy which can be achieved with slide rules of different lengths and on different parts of the scales.

Notes. (1) The statements that an angle is 17·50, that it is 17·5, and that it is 17½ degrees do not mean the same thing. The first statement, by convention, implies 4-figure accuracy—a result between 17·495 and 17·505; the second, 3-figure accuracy—a result between 17·45 and 17·55. The third statement is not so clear in its implication; it means we cannot tell whether the angle is nearer 17 or 18; if given as an answer to a measurement it might mean between $17\frac{1}{4}$ and $17\frac{3}{4}$: this can be put briefly as $17\frac{1}{2}$ within $\frac{1}{4}$.

---

* See also p. 72.

(2) A bad way of approximating to a small quantity is to find (approximately) two large quantities of which it is the difference; the weight of the captain's watch is not well found by estimating the ship's displacement before and after it fell overboard. An example of this is the evaluation of $\frac{1}{9{\cdot}84}-\frac{1}{9{\cdot}85}$ using 4-figure reciprocal tables:

$$\frac{1}{9{\cdot}84}-\frac{1}{9{\cdot}85}\simeq 0{\cdot}1016-0{\cdot}1015=0{\cdot}0001.$$

Here, although we worked to 4 significant figures, we have only got one significant figure in the result, and even this is not reliable.

We could, however, improve the accuracy while still using 4-figure tables by the following working:

$$\begin{aligned}\frac{1}{9{\cdot}84}-\frac{1}{9{\cdot}85}&=\frac{9{\cdot}85-9{\cdot}84}{9{\cdot}84\times 9{\cdot}85},\\ &=\frac{0{\cdot}01}{9{\cdot}84\times 9{\cdot}85},\\ &=0{\cdot}000\ 103.\end{aligned}$$

9·84	0·9930
9·85	0·9934
Den.	1·9864
0·01	$\bar{2}{\cdot}0000$
Den.	1·9864
0·000 1031	$\bar{4}{\cdot}0136$

In this way we have retained 4 significant figures in our result but the last figure is uncertain. On the other hand, the difference can be found directly by increasing the accuracy of the approximations using long division to express the reciprocals as decimals:

$$\begin{aligned}\frac{1}{9{\cdot}84}-\frac{1}{9{\cdot}85}&=0{\cdot}101\ 6260-0{\cdot}101\ 5228,\\ &=0{\cdot}000\ 1032;\end{aligned}$$

but here we have had to work to 7 significant figures in order to obtain the same order of accuracy as in the previous working, the last figure again being uncertain.

(3) British Standards recommend the use of the sign $\simeq$ for 'is approximately equal to'.* A connecting sign of this sort applies strictly to the two expressions connected by it,† so that we should write:

$$\pi\,.\,7^2\simeq\frac{22}{7}\times 7^2=154.$$

or

$$\begin{aligned}\pi\,.\,7^2&\simeq\frac{22}{7}\times 7^2,\\ &=154,\end{aligned}$$

from which we deduce that $\pi\,.\,7^2\simeq 154$. It may be helpful, in this example, to read the '=' sign as 'which is equal to'.

(4) It is a useful working principle that when an approximation is divided

---

* British Standard 1991: Part I: 1954, *Letter symbols, signs and abbreviations*, p. 14.

† See T. W. Chaundy, P. R. Barrett, Charles Batey, *The printing of mathematics*. (O.U.P.) 1954, p. 38.

by n (>1) the error is also divided by n. This is the basis, for example, of the method of finding the circumference of a cylinder by measuring the length of several turns of string. On the other hand multiplication of a measurement by n exaggerates the error. If a metre rule with an error of 0·01 cm were used to measure a kilometre, an error of the order of 10 cm might be expected. This principle also has applications to errors arising in the course of calculations. For example if $\frac{2}{3}$ of 0·5186 is calculated by first dividing by 3 to four significant figures and then multiplying by 2, the last figure of the answer has an error of 1. This error could have been avoided by first multiplying and then dividing. It can also be avoided by working to one more place than is required in the answer.

Another aspect of the use of approximations to which some attention should be given is the value of delaying the substitution of an approximate value. For example in calculating $3{\cdot}42\pi + 1{\cdot}85\pi$ it is simpler to write it in the form $(3{\cdot}42 + 1{\cdot}85)\pi$ before substituting a numerical value. Such work may well be taken with the use of brackets in algebra.

CHAPTER 14

INTRODUCTION TO STATISTICS

IF the members of a class measure the same quantity and the different answers are collected, it is natural to ask whether it is possible to derive an estimate more reliable than the individual measurements. Such work, which can begin quite early in the arithmetic course, provides an introduction to some of the elementary concepts of statistics, e.g. average or mean, and median.

For example, each pupil measures the length of his shoe and is then asked to measure the length of the classroom floor by a 'heel and toe' method, i.e. by counting the number of shoe-lengths. On one occasion the following measures were obtained and put on the board in order of magnitude: 19′ 4″, 21′ 1″, 21′ 7″, 22′ 1″, ..., 24′ 3″, 24′ 4″, 25′ 5″, 28′ 3″. There were 27 measurements in all. After some discussion the first one and the last two were struck out as being 'obviously wrong' and the average of the remaining 24 was found to be 23′ 1″. Then and only then, a tape measure was discovered and this yielded a measurement of 23′ 4″. The calculation of the average is lengthy and methods of shortening it will be welcome. A typical suggestion is to calculate the average of the excess of each measurement above 21 ft. If the measurements are arranged in order of magnitude it is natural to introduce the median (the central value or, if the number of measurements is even, the average of the two middle ones) as a more easily found central number. This in turn may suggest the advantage of taking a trial average near the median and calculating the average of the deviations by a process of cancelling as in bridge scoring.

$$\begin{array}{cc} - & + \\ \hline \cancel{2} & \cancel{12} \\ \cancel{8} & \cancel{7} \\ \cancel{6} & \cancel{3} \\ \cancel{0} & 2 \\ \cancel{3} & 2 \\ \cancel{4} & \cancel{5} \\ \cancel{2} & \cancel{7} \\ \cancel{1} & \cancel{1} \\ \cancel{6} & \\ \cancel{3} & \\ \hline & +4 \end{array} \qquad \text{Average} = \text{trial average} + \frac{4}{18}.$$

[Zero deviations are recorded to enable the total number of entries to be checked.]

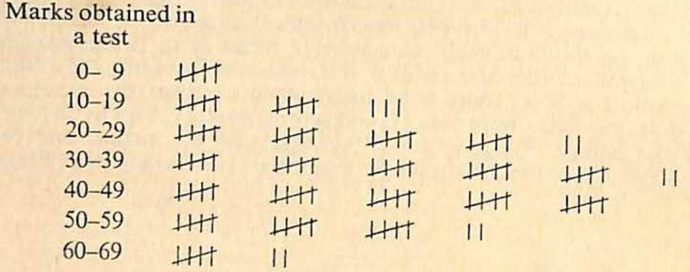
Similar exercises which have been found suitable in the first year of a secondary school are as follows:

(1) When introducing the metric system let groups draw lines of various lengths (1, 2, . . ., 6, 7 in.), measure them in cm, and estimate the length of 1 in. in cm. Collect results and calculate the average.

(2) Distribute maps of a region such as Wales duplicated on squared paper by means of the roller apparatus used in the geography department. Estimate the area of Wales by counting squares and determine the average and/or median.

(3) Let each pupil determine by means of measuring tapes in the playground the distance he walks in, say, 100 normal paces. Then determine an easily defined local distance, e.g. length of a football pitch, distance from school gate to station, etc. Determine the average. When asked for the 'right answer' look up the figures obtained in previous years, and perhaps consider how to find an average of averages. (The number of measurements as well as the average must, of course, be recorded each year.)

A second use of the average (or mean) and of the median as a convenient representative of a group of numbers must be introduced at some stage. This is the purpose of such numbers as average age, average mark etc., and this use occurs particularly when it is desired to compare groups of numbers. Other examples are average rainfall, average temperature, batting and bowling averages. Metereological averages can be related to local observations, and a recent copy of Wisden's is an excellent source of exercises on cricket averages. Statistical surveys, such as collecting travel times to and from school, stimulate interest and raise new problems in handling results. This is the place in which to introduce the ideas of class interval, frequency, and the graphical display of statistics. If the following recording system is used, the diagram itself is a rough histogram:



Throughout the work in statistics, diagrams of all kinds should be in constant use. Not only should they be drawn by the pupils but specimens of different types should be collected.

Artificial questions designed to correct exaggerated or mistaken ideas about the information conveyed by a mean value have their place. For example:

(i) In a test two groups of 10 pupils obtained the following marks out of 10:
Group A. 1, 4, 5, 5, 5, 5, 6, 6, 6, 10
Group B. Each pupil had 7 marks.
Group B has the better average but Group A contains both the lowest and the highest mark.

(ii) *A* has played 10 innings and has an average of 20. *B* has played 4 innings and has an average of 19. In the next match *A* scores 31 and *B* scores 30. What are now their averages? It surprises the pupils to find that *B*'s average is higher than *A*'s.

There will be opportunities for the discussion of the meaning of the statistics and the interpretation of the results of the calculations. For example, the accuracy to which an average is calculated depends upon the purpose to which it is to be put and upon the accuracy of the data.

A full discussion of the advantages of the median and mean cannot be made until the concept of standard error has been developed. This is because the most cogent reason for the more frequent use of the mean is that its standard error is less than that of other measures of central tendency. However, some discussion of the reason for having these two representative numbers is almost certain to arise from the fact that a choice has to be made between them; and the discussion will range wider still if the mode (see below) is mentioned, as it deserves to be. The *median* is sometimes quicker to calculate than the mean; for example, a form order of marks is required in many schools and, if this is available, the median can be found immediately. When there are a number of extreme values which would influence the mean, the median is likely to give a more typical value: and a similar situation arises when a distribution is markedly skew. Another case where the median is sometimes preferable is when a distribution is tabulated with one or both ends left open, e.g.

Number of subjects taken at O level of the G.C.E.	3 or less	4	5	6	7	8 or more
Number of candidates	20	12	23	35	21	17

Here the number of candidates has been given for each of the more central numbers of subjects and the median can be confidently stated to be 6; but in the distribution of marks above, while it is quite easy to discover that the median lies within the range 30–39 marks, its estimation to the nearest mark is an interesting exercise in interpolation. The mean has the advantage to pupils that it is familiar, and it is more easily defined with clarity. If $f_1, f_2, \dots$ are the frequencies of the measurements $x_1, x_2, \dots$, the mean is calculated from the formula $\frac{\sum f_1 x_1}{\sum f_1}$. There is no such simple formula for the median. From here it is easy to go on to point out that this makes the mean easier to deal with in theoretical work and that when calculating machines are available the computation involved in calculating the mean is not nearly so laborious as it is with pen and paper. The *mode* is the value of the quantity measured which occurs most frequently, e.g. 6 in the table above; but when the measurements are collected in classes, as in the rough frequency histogram above, it is better to refer to the 'modal class'; there the modal class is 30–39 marks.

At a later stage in the course, the question of the reliability of averages may be raised with profit. The early work on averages may have led to an uncritical acceptance of such statistics: a class will readily see that the average of their individual estimates of a length, of an area, of π, or of any measurement they may make, is likely to be more reliable than a single result. What may not be clear to them is the fact that the average itself would almost certainly have been different if the measurements had been performed by another class. At some stage in a pupil's education, he should be taught 'how to talk back to a statistic' (Darrell Huff. *How to lie with statistics*); at present many 'educated' people have very little idea of how to examine critically statistical evidence of even an elementary nature.

The pupils will probably have asked themselves such questions as, 'What is the average height of men (or women) in this country?' They may even quote a figure with disarming certainty: if so, here is an opportunity to sow doubt on the authority of the source. Anyway, such a problem, posed by teacher or pupil, can give rise to the problems of deciding what set of individuals to measure or question. This will be most effectively illustrated if the pupils undertake a survey themselves. Inside the school they may be interested in finding out the average number of bus journeys made per week or the average time spent on homework in the same period. Heights, weights, ages of pupils or parents, number of brothers and sisters, books read in a year, pocket money, size of shoe, rates of reading, etc. can be treated in the same way. Outside school, much will depend on the locality and on opportunities of individual pupils. The average amount spent by customers in a shop, consumption of electricity, rateable values, traffic densities, and the weight of a standing crop are examples of surveys which present problems of making the necessary observations.

When individuals or groups investigating the same population obtain different averages, doubt is cast on the reliability of the results. It is important to distinguish between the variation due to making observations which are not representative and the variation due to chance. The importance of the process of selection has been mentioned above: the variation due to chance needs more advanced mathematics for a full treatment and it is intended to provide a discussion of this matter in a future Report on the Teaching of Statistics in Schools. However, an example can give pupils some insight into random fluctuations of a mean. A simple classroom device is a pack of plain cards, numbered consecutively from 1 to 100, although the pupils are not told this yet. On each of a number of occasions, ten cards are drawn at random (but not replaced until all 10 have been drawn) the average is recorded, the cards replaced, and the pack shuffled. With several identical packs, the pupils can work in small groups and 50 averages obtained quite quickly. From such a pack of cards, fifty averages might be distributed as in the figure. We note that:

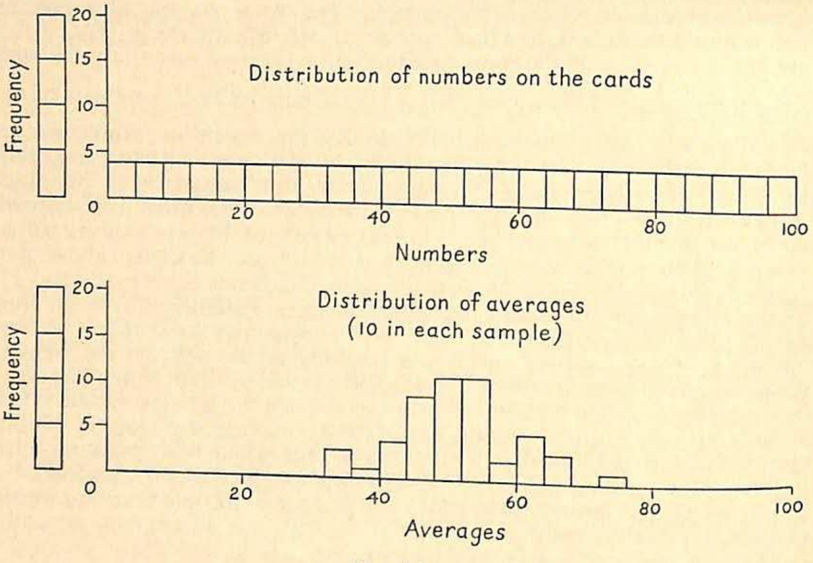


FIG. 14.1

(i) the averages appear to be clustered about the mean of the numbers in the pack;
(ii) the distribution is roughly symmetrical;
(iii) the averages are far less spread out than the numbers on the pack.

Since averages of groups drawn from the same population vary, it can be seen that we cannot know what reliance to place on an average unless we are given some indication of how much variation may reasonably be expected. Statements about averages in the press can be examined in the light of this knowledge but teachers should not allow the discussion to range outside their own knowledge of statistics.

CHAPTER 15

TRANSITION FROM THE PRIMARY SCHOOL

WORK attempted at the primary stage is changing rapidly both in content and in approach. There is a wide range of attitude, attainment and ability to be considered in entrants to all types of secondary school. It is however generally true to say that for the majority instruction in arithmetic has loomed large at early stages. A great many entrants have covered a good deal of the groundwork of computation and spent time practising rules often without acquiring any real mastery of the material involved, though perhaps gaining considerable facility in reckoning.

Since there is so much variation in the background of pupils entering the grammar school, some teachers have begun the mathematics course with prolonged revision. This is a mistake. It is important that a pupil entering a secondary school should get as much thrill as possible out of this new stage of his education. A mathematics course must certainly include opportunities for revision but, in arithmetic particularly, all the work is, or can be, revised when applications of it are made later on in the course. If more frequent revision is needed, this can be done in brief spells interspersed in new work.

In some cases a pupil has failed to assimilate (or even missed altogether) matter far behind the apparent growing point of his knowledge. Then a special approach is needed which may be called remedial treatment; this has been discussed at length in Chapter 4 of the Association's Report, *Mathematics in secondary modern schools*. The problem is one which arises more frequently in modern schools but every mathematics teacher in a grammar school should be ready to undertake this work whenever it is needed. The reader is also referred to the same Report for suggestions about assessing the outcome of mathematical education before the secondary stage. The relevant sections are: 2·0, 2·01 and 2·4 to 2·43.

It is important that the teacher of an entrant class should acquire as much information as possible about the attainments and attitudes of his pupils. Some of this may be obtained from primary school record cards, or from contact with the primary school teachers and much can be gathered from conversation with the pupils. However, children often talk about an idea without having developed a correct concept of it. This happens in infancy when children use number words without understanding their meaning and it goes on, unless the teacher is very careful, right through the school course of mathematics. In the field of mensuration, particularly, children often arrive at the secondary school with only partially developed concepts of area and volume. Some suggestions for work to help such pupils are given in Chapter 9. It is not suggested that a teacher should refrain from making progress in a particular field because the relevant concepts are undeveloped. Rather, the concepts are developed when the pupil acquires experience in that field; but at present work designed to help pupils in this way (as opposed to simply teaching computational techniques) is not as common as it should be.

'Stage A' geometry was described in the Association's Report, *The teaching of geometry in schools* (1923). The work mentioned in the previous paragraph might well be described as 'Stage A arithmetic'. Its main purpose is not so much to communicate knowledge as to help our pupils to understand the concepts with which arithmetic deals.

CHAPTER 16

SOME SUGGESTIONS ON CLASSROOM PROCEDURE

THE first stages in teaching a new topic are particularly important since first impressions endure and may permanently influence a pupil's thinking. It is generally recognised that simple examples are best, chosen not only to ensure successful working out by the pupil, but more importantly so that the mathematical thinking is not obscured by the intricacies of the calculation. Special cases may be misleading in that they do not give the pupil a reasonable view of the topic. It is for this reason that in the section on graphs in this report we suggest beginning with a few curved graphs so that the general technique of plotting points and joining them with a smooth curve is met before the special technique of using a ruler when the graph happens to be a straight line. Simple proportion, again, is only one kind of functional relationship and should be put in perspective early by giving examples of other kinds, even though it may not be possible to do much with them. The examples should gradually become more complex and too much stress should not be laid on those which start from exact data or 'come out' exactly, lest quite a wrong idea is obtained of the way in which mathematics is applied in real life.

The teacher has, broadly speaking, two lines of approach. The first is traditional and familiar: it is teacher-centred and relies on exposition and examples worked on the blackboard, but the pupil learns largely by listening. In the second the emphasis is on the pupil's activity: the teacher's task is to contrive the appropriate stimulus to learning, sometimes by posing a challenge or problem and guiding the pupil to discover a solution, at other times by suggesting an activity, e.g. the use of a calculating machine, leading to the discovery of a generalisation. This method of 'guided learning,' as it has been called, has always been used by some teachers in presenting any subject and is now becoming more acceptable in mathematics through the pioneering work of John Perry and Sir Percy Nunn, with its successful implementation in our generation in infant and junior schools. Essentially one begins with a problem (The recipe says 1¼ lb of sugar to 1 lb of fruit: we have 3½ lb of fruit. . . : what can we do to answer this? . . . One refrigerator has these dimensions . . . , another these . . . : how can we compare the two? . . .) and uses the natural curiosity and intelligence of the children to evolve a solution. In so doing, it is important for them to be ahead of the teacher in their thinking and reckoning (as in doing riders in geometry when they draw their own diagrams before any diagram is drawn on the board by the teacher). A more advanced instance of this approach is that of F. C. Boon who used to encourage pupils to discover how a table of logarithms worked by getting them to compare the results of very simple operations with entries in the table. After some guided problem-solving of this kind, a standard pattern of thinking and operating will emerge and this can be summarised on the blackboard and noted by the class: after which elaborations may be developed, but fundamentally children are thinking and doing for themselves.

They can for instance discover for themselves that the distance round the boundary of a circle is about three times its diameter. This is likely to lead someone to ask how the ratio can be found more accurately. The teacher can invite suggestions and possibly contribute one of his own, and set different groups to follow them up. During the resulting activity the pupils are compelled to think about the relationship of diameter to circumference. They have time to take in its implications and to understand how it fits into their existing knowledge. When the results of the group activities are pooled the whole class is likely to have a clearer understanding of the significance of π than if it had been given a definition and a set of exercises to work. Such a method, of course, takes time and is often slower than the old-fashioned talk-and-chalk, but it can be enormously rewarding for teacher and pupil if real understanding is achieved. No doubt as calculating machines and mathematical laboratories become more common in schools pupils will have more opportunities of learning for themselves how things work as if they were real craftsmen or researchers. And then it may be necessary for teachers to guard against the danger of proceeding too slowly. Obviously the pupil has not sufficient time to retrace the researches of the ages—and the teacher's role must be to shorten the process by suggestion and systematisation. His task is not to expound to a passive class, but to let a new principle or generalisation crystallise as quickly as possible from the children's own thinking. An excellent illustration of this method in action will be found on p. 46 of *Mathematics Teaching* No. 23, Summer 1963.

Eventually the general principles of the subject and the rules of thumb must emerge but here, perhaps more than anywhere else, the teacher will do well to remember the old motto 'more haste, less speed'. Premature introduction of a rule of thumb usually leads to subsequent breakdowns of memory, and tedious bouts of revision. It is better to delay the final generalisation until it is almost spontaneously suggested by the pupils themselves. It is vitally important to give a clear justification for any rule-of-thumb; some pupils will understand it, a few will remember it, but all should realise that there is a justification and that it is logic and not magic.

On the other hand, it is most important that this final crystallisation of a fluid method into a hard-and-fast rule should be well defined. The pupil must know and use such rules-of-thumb, and the habit of using them must be developed. Mathematics essentially aims at an economy of thought, and the more operations a pupil can perform without having to stop and think, the more he will learn about the subject. 'Civilisation,' as Whitehead says, 'advances by extending the number of important operations we can perform without thinking about them.'

Ultimately the basic facts and rules of arithmetic must be known so thoroughly that they can be recalled without conscious effort. This takes time and patience, but one method by which the process can be helped is the setting of a few quick questions at the beginning of the arithmetic lessons for as long as may be necessary. These questions should be easy so that most of the class get them right, and not too numerous so that they do not encroach too much upon the lesson. Their aim is simply to recall repeatedly the fact or process it is desired to learn. It may be the number of square yards in an acre, or the decimal equivalents of simple fractions such as $\frac{3}{8}$, or the division of fractions, e.g. $\frac{3}{4} \div \frac{2}{3}$. There is no need to restrict ourselves to arithmetic, and the question just quoted might well be followed by something like $\frac{3}{a} \div \frac{2}{b}$, or $\frac{3}{x} \div \frac{2}{x}$. Apart from the value of the repetition, these quick questions can be made into a kind of game, and then they can give the pupil the impression that the teacher, for his part, is doing all he can to help him to learn. This is so much better than throwing the whole responsibility on the pupil with the remark, 'Learn this for homework. I will test you tomorrow.'

But, of course, the staple diet of the classroom consists of longer written exercises. These are needed to give experience and to consolidate the grasp of a new idea. Care should be taken with the choice of examples and there are several pitfalls. While consolidation requires the working of a number of examples, this is sometimes carried to excess. On the one hand, success with a number of similar examples gives a pupil confidence; on the other hand, repetitive exercises will tend to associate a new process with a particular type of example and will become boring. The teacher therefore needs to draw a balance with each particular class between examples which give confidence and those which will challenge and stimulate.

When examples are set from a text book, care should be taken to check that they are well graded and suitable for the class. Examples which are not taken from the text book have a freshness about them which the printed ones lack. Every effort should be made to bring the teaching in the classroom into close relation with the life of the child outside it and information brought by pupils on their own initiative should be used if possible. Some may have parents or friends with specialised knowledge and can bring in data of particular interest. Anything with a numerical aspect which interests the class is a valid basis for a piece of arithmetic. The best exercises are those which arise out of questions asked by the pupils themselves and occasionally it may be worthwhile inviting them to make up their own exercises, some of which may be set to the rest of the class. But there is one difficulty in the use of real-life examples—the computation is often so heavy that it distracts attention from the principles involved. One way of dealing with this difficulty without losing too much interest is to make a suitable approximation. For example, a young boy brought to a mathematics lesson the information that Zatopek ran 10 000 metres (6 miles 376 yards) in 29 min 2·6 sec. This led to, and gave added interest to, the simple question, 'How fast is a man running when he covers 6 miles in 30 minutes?'

Examples of the type we are considering here must be written down, and almost immediately the teacher is confronted with a dilemma: should he require the accurate use of words and symbols from the start, so as to lay the foundations of good habits and prevent bad ones from being acquired, or will such a course tend to make the work seem artificial to the pupil?

Feeling for style and accuracy of expression, and for form in the presentation of work, are of gradual growth. Children use a small vocabulary for ordinary talk, not only because they have a limited stock of words but because of their limited experience and their ignorance of the finer shades of meaning. They use words loosely and their thought is unorganised; and this is all quite natural. It is the task of the teacher, in any subject, to show them how to think more clearly and how to express their thoughts on paper.

In teaching mathematics there is a danger of demanding too much in this way at an early stage. Accuracy of expression may be always desirable, but only good judgment can determine how much should be required at any given time. To decide this question the teacher must keep his hand on the pulse of the class; interest will flag when there is too great an insistence on what is, to the immature mind, merely detail.

A good style in the presentation of mathematical work is a valuable aid to accuracy and leads to economy of thought and time. Good style consists in telling the truth simply and clearly. It is the product of clear thought and thus often the index of ability; but also it can be taught. Bad style is the result sometimes of failure to understand; but more often of indolence or lack of teaching. It is a trial to the reader's patience and a handicap to progress.

The necessity for clear expression is not apparent to the beginner; nor can he be left to learn by experience. *He must be shown on the blackboard* what good style means and must be encouraged to acquire good habits. Any effort he makes towards clear expression will bring its reward of increased accuracy and comprehension, enabling him to tackle larger scale work as time goes on.

A well-written piece of arithmetic is more easily checked by reading through than by doing the work again. It consists normally of sentences, accompanied by 'sums' or 'working'. The 'working' is tabulated, but to the sentences the usual rules of grammar apply. A good test is to read the sentences aloud. The '=' sign is a verb, and '∴' is a conjunction. The working must never be hitched on to the sentences, e.g.,

$$\text{`The number of boys per bus} = \frac{230}{9} = \frac{9\,)\,230}{25{\cdot}56}\,,$$

nor yet relegated to a 'rubbish-heap' margin; sometimes it should be placed in the middle of the page, more often at the side, opposite the sentences to which it is relevant. It should consist normally of operations on numbers, the units being dealt with in the course of the sentences; and the numbers should be written full size and arranged neatly in columns.

Mis-use of the '=' sign should not be tolerated: e.g. '3 horses = £80', '50 per cent = £26', or (in a question about a doctor's practice) 'losing patients = 20 per cent'. A scale should be stated as '1 inch repr. 2 miles' or '1 in. : 2 miles'. If x is used, it should represent a number, not a sum of money or other quantity. A solution should not begin 'Let x = time', nor finish '$\therefore\ x = \frac{2}{5}$ h = 24 min'. The answer should be written as a sentence: 'Time taken = 24 min.' The writing of 'Answer = 24 min' should be discouraged.

Good style is more easily caught than taught, and the teacher's own style on the blackboard will have more influence than precept, reward or punishment. But direct teaching is necessary as well. Children can be taught at an early stage to use their limited vocabulary effectively and to build up the habit of clear expression. The writing of such simple sentences as 'Cost = ...', 'Area = ...', 'Time taken by Smith = ...' makes no great demand on their patience, but goes far to give them the idea that what they put on paper should have meaning. The practical-minded pupil who says 'It gets the answer right and that's all that matters' can be induced to admit that 'we ought to say what we're talking about'.

This is not to say that there is no place for paper jottings. There are many occasions in everyday life when the backs of envelopes can be usefully pressed into service, and in the classroom too it is not every piece of work that should be fully written out. Sometimes the teacher, perhaps when explaining something new, may want the class to share in the work by doing some of it for themselves on rough paper. Some work again is best arranged in tabular form.

The indifferent style which is sometimes adopted by pupils may be due to their keenness; their thoughts are far ahead of their pens. But it may also be due to their being given work which is too easy or too difficult; in the first case the writing of logical statements is apparently superfluous, in the second insistence on good style may be the last straw which makes the whole task seem too formidable to be tackled.

A caution must be added: teachers, usually those with some experience, may require perfection of style too soon. This may have the effect of substituting the mental operations of the teacher for those of the pupil. A mean has to be observed between the toleration of slipshod work and a requirement of such precision as can only be satisfied by dictation. To each age corresponds its own vocabulary and its own standard of exactness of thought and expression. Every traveller on the continent knows how a limited vocabulary cramps his conversational style, and anyone who explores unfamiliar mathematical ground knows that understanding of a vague kind must be followed by a further effort before the new idea can be expressed clearly on paper.

The difficulty of clear expression must not be under-rated, but neither must its value. The pupil may be right when he regards the good presentation of the work as an additional task, but he must be brought to see that the effort is worthwhile.

One of the attractions of mathematics is the relative certainty of the conclusions reached in it. Pupils are always pleased to tick their answers right or to see the teacher's ticks, and the greater the proportion of answers right, the greater will be the enthusiasm of a class for arithmetic—provided that the questions set are not so easy that the pupils think them a waste of time. Checking is an important aid to accuracy and is not a part of teaching that can be ignored with impunity.

First, it should be stressed that checking without the right attitude to it is a waste of time. Pupils making good progress in mathematics want to get their work correct and should see that checking is a means to this end. A check that is done because the teacher requires it, and not as a help to getting the answer right, is useless. The object should be to encourage the habit of checking. There are two questions that pupils should ask themselves on completing a piece of arithmetic: 'Is it a reasonable answer?', 'Is it right?' Ways of trying to answer these questions are discussed in the paragraphs below.

Perhaps the most important thing for a pupil to check is that he has done what he meant to do. This may be encouraged by giving answers in the form: 'The volume is 36 cu. in.' or 'The surface area is 5½ sq. yd,' rather than: 'Answer 36 cu. in.' or 'Ans. 5½ sq. yd'. Next he should ask himself, 'Is this a reasonable answer?' If, for instance, a car going from London to Winchester has an average speed of 325 miles per hour, or a pound of sugar costs ½d., he should see that these are not consistent with everyday experience. In this respect text-books should not be too out of date, although it is interesting for the pupils to compare pre-war prices with those of the present. It will be seen that a written check cannot be expected in this comparison with everyday life, but some answers to the question, 'Is it reasonable?' can be written down. For instance (see Chapter 12) logarithms are used to evaluate an expression and the answer 29·1 is obtained. A rough approximation to check the position of the decimal point can be written down as follows:

$$\frac{18{\cdot}7 \times 15{\cdot}8}{216 \times 0{\cdot}047} \simeq \frac{20 \times 15}{200 \times 0{\cdot}05},$$

which is easily evaluated. In this particular case, the lower streams will find the idea of approximation difficult and it is well to bear in mind the principle that any check should be simpler to the pupils than the calculation that is being checked. Pupils who can follow a routine method, but find it hard to think for themselves do not take easily to rough approximations. It can be useful to suggest that approximations to one significant figure are sufficient.

In considering the ways of answering the question, 'Is it right?' it is well to think what mathematicians and pupils do when they really want to be sure of getting a correct result. One of the most frequent ways is to compare the answer with someone else's answer to the same question. This is what is often done in adult life and what pupils do with their homework or prep (and sometimes in class). Again, this is what the teacher and pupils do when an answer book is used. If no one else's answer is available, the best method of checking is to work the question again by a different method. This is what is done in checking addition by adding up and down a column of figures. Another method of checking an answer is to use that answer to deduce something that is already known, for instance, the errors in the subtraction

$$\begin{array}{r} \bar{1}\cdot3043 \\ \bar{2}\cdot6859 \\ \hline 1\cdot6284 \end{array}$$

will easily be detected by addition. Similarly division may be checked by multiplication. An advantage of checking by a different method is that it eliminates the very real risk of making the same mistake twice. Readings from tables should be checked—comparing what has been written with the entries in the tables to detect miscopying.

Practical work introduces calculations to which the answers are not verifiable by looking at the back of the book. They can be used to introduce the idea of checking by the consistency of several results. The area of a triangle, for example, can be found by measuring a side and an altitude in three ways, or the different calculations of a height or distance made by the members of a class can be compared.

Inaccuracy often is the result of untidy work and slovenly figures, and finding a mistake can be made much more difficult if part of the calculation is crammed into a small space. The teacher's own attitude to tidiness and checking will make an important contribution to the accuracy of the pupils' work. If marks are given for quantity, it will generally pay to do a large number of questions without checking and with little regard for tidiness. But above all, the teacher should bear in mind that the educational process involved in checking is only complete when the pupil habitually applies checks that will make him fairly sure that his work is correct for his own satisfaction.

But even if this ideal is realised there will still be mistakes and we have to consider what to do then. Before deciding upon the traditional plan of setting 'corrections' the teacher should have a clear idea of the purpose he is trying to serve. Mistakes in arithmetic may be divided into two classes: those in which the pupil has not applied some principle correctly and those in which some mechanical mistake has been made, frequently inaccuracy in the use of tables or in simple addition or subtraction. From the work alone, it is sometimes difficult to decide into which class a particular error falls: it is the pupil's understanding of the process which is the determining factor. In the first case corrections or, better still, similar examples may be set. In the second case corrections may produce the opposite effect to that which is desired. If a child is bored with the work, he will not take such care over it and further mistakes will probably follow. Too many corrections slow down the pace, and interest in the subject may be lost.

Before corrections can usefully be done it is necessary for the pupil to find out or be told where he has gone wrong. In a large class this may take a long time, and it may be useful to let some of the better pupils act as group leaders and give help to the members of their own group. The ideal is to give the least amount of help which will enable the pupil to put right what is wrong. It may be 'That's wrong. Try and find the mistake yourself and put it right', or it may be 'The mistake is here: go and correct it', or it may be a series of questions which end with the pupil saying, 'I see. How silly of me.' This ties up with checking since, if the helper does nothing but ask questions, it may be pointed out that the pupil could have found the mistake by himself, if only he had thought of the questions.

When new pupils in a secondary school make mistakes it is tempting to lay the blame on the methods taught in their old school, where these differ from those favoured in the new school. But before a teacher attempts to alter the habit either of work arrangement or of a thought process of a pupil, he should consider carefully his reasons for doing so. Uniformity of method within the class makes for ease in class teaching, but this is not, in itself, sufficient reason for attempting to change the habits of any pupils whose methods may differ from those of the majority. A habit, once formed, may be difficult to alter, and can only be altered with the co-operation of the pupil. If the method employed is giving good results no attempt should be made to change it unless the teacher is sure that it will ultimately prove a hindrance to progress in later work.

The more able the pupil the more flexible he will be. When the use of logarithms is being taught, and the need to subtract numbers with negative characteristics arises, the teacher must be sure which method of subtraction is being used by each pupil. At this stage it is not unknown for a child who has been taught decomposition, voluntarily to change to equal additions, or complementary addition, and to persist with the new method in all future work. No attempt should be made by the teacher to compel such a change against the will of the pupil, or confusion will certainly result.

It is desirable, whenever possible, to teach in the first instance the arrangement of work which will be acceptable in all future work. Habits which tend to economy of thought should be introduced when the pupil appears ready to receive them, and should be encouraged but not enforced.

APPENDIX ON UNITS

THE *Weights and Measures Act* 1963 defines the yard as 0·9144 metre exactly and the pound (mass) as 0·453 592 37 kilogramme exactly. The metre, kilogramme, litre, ampere, ohm, volt and watt are defined internationally and this country adopts the international definitions in force from time to time.* The Act defines the gallon as 'the space occupied by 10 pounds weight of distilled water of density 0·998 859 gramme per millilitre weighed in air of density 0·001 217 gramme per millilitre against weights of density 8·136 grammes per millilitre'.

The units of measurement of length, area, volume, capacity, mass and weight which are permitted to be used in trade are set out in Schedule 1 of the Act. They are listed below:

Imperial units	*Metric units*
Measurement of length	
Mile	Kilometre
Furlong	Metre
Chain	Decimetre
Yard	Centimetre
Foot	Millimetre
Inch	
Measurement of area	
Square mile	Hectare
Acre	Dekare
Rood	Are
Square yard	Square metre
Square foot	Square decimetre
Square inch	Square centimetre
	Square millimetre
Measurement of volume	
Cubic yard	Cubic metre
Cubic foot	Cubic decimetre
Cubic inch	Cubic centimetre
Measurement of capacity	
Gallon	Hectolitre
Quart	Litre
Pint	Decilitre
Gill	Centilitre
Fluid ounce	Millilitre
Bushel	
Peck	
Fluid drachm	
Minim	

---

* The current definition of the metre in April 1964 was 1 650 763·73 wave lengths in vacuo of the orange-red line of krypton-86. In October 1964 the second was redefined as 9 192 631 770 oscillations of the caesium-133 atom.

Measurement of mass or weight

Ton
Hundredweight
Cental
Quarter
Stone
Pound
Ounce
Dram
Grain
Ounce troy
Pennyweight
Ounce apothecaries
Drachm
Scruple

Metric ton
Quintal
Kilogramme
Hectogramme
Gramme
Carat (metric)
Milligramme

Measurement of electricity

Ampere
Ohm
Volt
Watt
Kilowatt
Megawatt

The bushel, peck and pennyweight cease to be lawful units five years after the Act comes into force.

The fluid drachm, minim, the ounce apothecaries, drachm and scruple will cease to be lawful units at some later date.

The carat (metric), the ounce troy, the ounce apothecaries, drachm, scruple, fluid drachm and minim are restricted in their use.

Amongst the lesser known units in general use,

1 rood = 1210 square yards,
1 are = 100 square metres,
1 gill = $\frac{1}{4}$ pint,
1 fluid ounce = $\frac{1}{20}$ pint,
1 cental = 100 pounds,
1 dram = $\frac{1}{16}$ ounce,
1 grain = $\frac{1}{7000}$ pound,
1 metric ton = 1000 kilogrammes
1 quintal = 100 kilogrammes
1 carat (metric) = $\frac{1}{5}$ gramme

Schedule 2 of the Act describes the existing United Kingdom Primary Standards and Authorised Copies thereof.

INDEX

References are to pages. Where there are two or more references under one heading, bold type indicates a principal reference.

Abacus, 9
Accuracy, 45, **74–78**
  of expression, 86
  of tables, 72
Addition, 13
  of decimals, 24
  of fractions, 21
Algebra, 52, 56
  laws of, 23
Antilogarithms, 69
Approximations, 74–78
Arrangement of work, 69, 90
Area, 16, **45–52,** 91
Averages, 79–82
Axiomatic approach, 4

Binary scale, 3, 4, **10**
Books, 12

Calculating machines, 3, 4, 10, 24, 71, 81
Cancelling, 23
Capacity, 48, 91
Checks, 88
Circles, 49
Classroom procedure, 84–90
Complex numbers, 7, 8
Compound interest, 41
Compound proportion, 38
Compound units, 15–19
Computation, 14
Cones, 50
Continued fractions, 64
Contracted methods, 1
Corrections, 89
Counting squares, 46, 49
Cylinders, 49

Decimal places, 45, 75
Decimal point, 27
Decimalisation, 18
Decimals, 20–28
Decreases, 31, 39
Degree of accuracy, 15, 45, **74–78**
Depreciation, 41
Dimensions, 52
Direct proportion, 38
Directed numbers, 7, 8
Discount, 30
Division, 13, 14
  compound, 18
  galley method, 10
  of decimals, 26
  of fractions, 22
  scratch method, 10

Electrical units, 92
Estimation, 44
Examples, 86

Four rules, 13, 14
Fractions, 7, **20–28**
Frequency tables, 80
Functionality, 39, 56

Graphs, 34, 43, **54–57,** 59, 65, 70–73
Groups, 4
'Guided learning', 84

Habits, 90
H.C.F., 11

Inaccuracy, 89
Increases, 31, 39
Index numbers, 32
Indices, 65
Inequalities, 3
Integers, 7
Inverse problems, 39–41
Irrational numbers, 7

Laws of algebra, 23
Length, 44, 91
  of curves, 45, 49
L.C.M., 11
Local authority finance, 42
Logarithms, 65–73

Man-hours, 38

Maps, 36, 46
Mass, 92
Matrices, 4
Mean and median, 79–82
Mensuration, 44–53
Metric system, 16, 44, 91
Minus sign, 5
'Modern mathematics', 3–5
Money, 15
Multiplication, 13
  compound, 16–18
  grating method, 9
  of decimals, 25
  of fractions, 21
  tables, 9

Natural numbers, 7
Numbers, 6–12
Numerals, 6

π, 49, 75

Parallelograms, 48
Patterns of numbers, 4, 6
Percentages, 29–32
Place-value, 4, 14, 24
Practice, 16
Primary schools, 83
Prime factors, 11
Prime numbers, 11
Profit and loss, 30, 40
Proportion, 34–38
Pyramids, 50

Rate, 33–43
Rates, 42
Ratio, 33–43
Rational numbers, 7
Ready reckoners, 17, 34, 38
Recurring decimals, 25
Roman numerals, 6
Rough estimates, 27, 88
Rules-of-thumb, 85

Scale drawing, 38
Scales of notation, 4, 10
Sets, 4, 30
Sequences, 4, 11
Significant figures, 45, 75
Simple interest, 40
Slide rule, 4, 38, **72,** 76
Speed, 42
Spheres, 50
Square roots, 58–64
  Newton's method, 60
Squares, 58–64
Standard form, 26, 67
Statistics, 79–82
Style, 86
Subtraction, 13
  of decimals, 24
  of fractions, 21
Symbols, 3–5, 7, 8, 77

Tables, multiplication, 9
  of logarithms, 70
  of squares and square roots, 64
  of weights and measures, 16
Trapeziums, 48
Travel graphs, 57
Triangles, 48
Trigonometry, 38

Unitary method, 35, 37
Units, 15, 16, 44–48, 52, 74, **91–92**

Variation, 39, 56
Volume, 16, **48–52**, 91

Weight, 15, 92
  American units of, 16

24/77

A second report on the teaching of arithmetic in schools

A report on the teaching of
arithmetic to the more able pupils
in secondary schools
prepared for

THE MATHEMATICAL ASSOCIATION

London: G. Bell & Sons Ltd

This book was taken from the Library of Extension Services Department on the date last stamped. It is returnable within 7 days .

Rs 6·30

A SECOND REPORT ON THE TEACHING OF ARITHMETIC IN SCHOOLS

Prepared for The Mathematical Association

A report on the teaching of arithmetic
to the more able pupils in secondary schools

INSTITUTE OF EDUCATION FOR WOMEN
2477
Dept. of Extension
SERVICE.
30.3.67
CALCUTTA-27

G. BELL & SONS, LTD
Portugal Street, London, W.C.2
1964

© The Mathematical Association
First published 1964

In 1958 the Teaching Committee agreed that the Report on *The teaching of arithmetic in schools* was in need of revision. A sub-committee was set up which reported to the Teaching Committee in March 1964. The revision necessary proved so extensive that the present report may be regarded as a new one. The members of this sub-committee were as follows:

Miss L. D. Adams
Mr W. Armistead
Mr J. K. Backhouse (*Secretary* from 1960)
Miss H. Bromby
Mr C. T. Daltry (*Secretary* to 1960)
Mrs H. L. Fudge
Mr L. G. W. P. Jones
Mr E. H. Lockwood
Mr C. G. Nobbs
Mr J. S. Peel (from 1960)
Mr G. K. Priestman (to 1960)
Mr A. W. Riley (*Chairman of the Teaching Committee*)
Mr T. R. Theakston (*Secretary of the Teaching Committee*)
Mr C. O. Tuckey (*Chairman*)
Mrs E. M. Williams

The printing of this Report follows the recommendations of British Standard 1991: Part 1: 1954, *Letter Symbols, Signs, and Abbreviations.*

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD
THE UNIVERSITY PRESS, GLASGOW

CONTENTS

CHAPTER PAGE

1. INTRODUCTION - 1
2. NEW DEVELOPMENTS IN SCHOOL MATHEMATICS - 3
3. NUMBERS - 6
4. SOME REMARKS ON THE FIRST FOUR RULES - 13
5. COMPOUND UNITS - 15
6. FRACTIONS AND DECIMALS - 20
7. PERCENTAGES - 29
8. RATIO AND RATE - 33
9. MENSURATION - 44
10. GRAPHS - 54
11. SQUARES AND SQUARE ROOTS - 58
12. LOGARITHMS - 65
13. APPROXIMATIONS - 74
14. INTRODUCTION TO STATISTICS - 79
15. TRANSITION FROM THE PRIMARY SCHOOL - 83
16. SOME SUGGESTIONS ON CLASSROOM PROCEDURE - 84

APPENDIX ON UNITS - 91

INDEX - 93

INSTITUTE OF EDUCATION FOR WOMEN
Dept. of Extension
SERVICE.
CALCUTTA-27

CHAPTER 1

INTRODUCTION

SINCE the publication of the last Arithmetic Report in 1932 there have been considerable changes in the background to the teaching of mathematics in schools.

Public opinion has become aware of the importance of mathematics in the life of the country and considerable thought has been given to methods of training mathematicians. Syllabuses have come under criticism, and much has been done towards the integration of the traditional branches of the subject.

The earlier report was written at a time when arithmetic was thought of as a separate school subject rather than as a part of elementary mathematics, when the use of tables of squares and square roots was rare, and when the farthing was still a coin to be reckoned with. Much attention was then given to computation and there was controversy among teachers as to the best methods, for example, in subtraction and in multiplication and division of decimals. Since then, opinion among teachers has hardened on some questions of alternative methods, which makes their discussion unnecessary. Contracted methods have been swept aside by the universal use of tables; and the whole subject is now viewed in better perspective as a part of elementary mathematics.

There has also been a change in the method of approach. In the early part of the century, as soon as new types of number such as fractions or decimals had been introduced, it was customary to teach the application to them of the 'four rules'—addition, subtraction, multiplication and division—for the sake of mastering the technique, somewhat apart from the possible use of this technique in cases likely to arise. But nowadays the view is that the problem comes first and the technique for solving it second. A new situation is presented as a challenge and time is allowed for pupils to attempt a solution. This is thought to be worth while no matter how fumbling their efforts may be.

This report discusses the teaching of arithmetic, as part of a general mathematics course, from the end of the primary school stage to the time when the General Certificate of Education is taken. It is written mainly for teachers of boys and girls of good ability such as are found in grammar schools and in the grammar school streams of comprehensive schools. There are many other pupils who are still not ready at 11-plus to begin such a course as is here envisaged, and appropriate guidance for their teaching may be found in the Association's Report, *Mathematics in secondary modern schools*.

In the present time of flux, it is not possible to anticipate what arithmetic will be included in school courses of the future. **So the inclusion of a particular example or topic in this Report must not be interpreted as a recommendation that these should necessarily be taught in schools.** Nor should the reader expect to find a comprehensive set of examples covering all the 'types' met with in arithmetic: a mathematical education does not consist of teaching pupils to tackle examination questions so much as to apply their knowledge of mathematics to situations both familiar and unfamiliar.

The time spent on arithmetic is nowadays likely to be shorter than it used to be and it is necessary to think of it always as part of elementary mathematics. But the importance of arithmetic remains. The science of number is a part of our culture and an ability to understand and use number is a fundamental social need. For many children there is a fascination in numbers and their properties; for all of them arithmetic is the foundation stone of mathematical knowledge. Its everyday applications are of interest to them and they like to feel they are learning something useful. But in fact they are doing more than this: they are beginning mathematics. Arithmetic leads on to algebra; particular numbers give way to variables, particular calculations to formulae, and the subject develops, as it did historically, through algebra and trigonometry to calculus and coordinate geometry. Through all of this, arithmetic is the pupil's sheet anchor, for by examining numerical cases he tests and clarifies his ideas and discovers his mistakes. The new topics which are now beginning to be taught in schools under the name of 'modern mathematics' have likewise their arithmetical side.

Finally it may be observed that arithmetic is the means by which most mathematics is applied, for the most general theory can only be brought to bear through the arithmetic of a particular case. In the world of applied mathematics numerical work is being pursued on an ever widening scale with the aid of machines and computers. This not only emphasises the importance of arithmetic but suggests that the lengthy calculations which were formerly undertaken in schools should now be omitted.

Arithmetic should be taught, from the start, with these ends in mind. This means, among other things, that the understanding of a process must be regarded as more necessary than the learning of a mechanical rule that produces the right answer; that in achieving this understanding accuracy and clear expression are vital; and that lengthy calculations should be avoided as likely to delay the grasp of underlying principles. It means too that a start on other branches of mathematics need not be unduly delayed; these give first-hand experience of some of the purposes of arithmetic besides being themselves the development to which arithmetic naturally leads.

CHAPTER 2

NEW DEVELOPMENTS IN SCHOOL MATHEMATICS

IF arithmetic is the beginner's key to future progress, it should be taught in such a way that it leads on to the rest of mathematics, both elementary and advanced. This means not only algebra and trigonometry, coordinate geometry and calculus, but also the new topics now being introduced into school syllabuses—the binary scale, calculating machines, sets, groups, matrices, statistics, probability, and so on. As teachers we have to be aware of the far horizon but without losing sight of the ground under our pupils' feet. We must therefore take notice of the new developments and the experiments being made in widening the syllabus and should ask ourselves how arithmetic can prepare the way for this new work and take part in its introduction.

It may be thought that the function of arithmetic is to open the door rather than to begin the conduct of the tour, and certainly much can be done to provide the experience which will enable the new ideas to take root more easily. As an illustration we may consider the place of inequalities in elementary teaching, It is now realised that there has been too much emphasis in schools on equalities and not enough on inequalities. It has been said that mathematics begins with inequality, for counting and measuring are preceded by a recognition that some quantities are greater than others. A related notion of great importance is the idea of betweenness, and this has too often been neglected. For example, if a class is asked the value of π, there are sure to be some who say 'It is $3\frac{1}{7}$'. The answer 'π is between 3·14 and 3·15' is no more difficult and a much better preparation for later work. At all stages of the school work, examples can be found which use the idea of inequality and prepare the way for its more formal treatment later on. Some examples of this sort are:

1. In a rectangular park $\frac{1}{4}$ mile long and 200 yards wide there is a bandstand at the centre and the band is audible for a distance of 75 yards. Find the area of the park in which the band is inaudible.
2. A savage dog is tethered by a chain 12 ft long to a post 8 ft away from a straight path. For what distance along the path is one in danger?
3. Give some numbers whose square roots are between 5 and 6.
4. (*a*) If 144 is expressed as the product of two unequal factors, why is one less than 12 and the other greater than 12?
   (*b*) It is desired to test whether 149 is a prime number. If it is found to have no divisor less than 12, why is it certain that it is prime?

The comparison of fractions is perhaps the most obvious and natural occasion for the consideration of inequalities, and may be used for the introduction of the inequality signs. There may be different opinions as to when these signs should be introduced. Children adopt new symbols with enthusiasm and enjoy finding opportunities for their use. On the other hand, the important thing is that they should use inequalities readily in their thinking, and this may be encouraged more by using the phrases 'greater than', 'less than' and 'between' for some considerable time.

Another basic idea is the distinction between a proposition and its converse.

The symbol '$\therefore$' is often used by pupils in an indiscriminate manner and there are occasions when attention can be directed to its 'forward' direction. This can be emphasised by the use of the symbol $\Rightarrow$ (which should be read as 'implies' or 'only if') and the contrasting sign $\Leftrightarrow$ (read as 'implies and conversely' or 'if and only if').

Thus 'Area of square = 1089 sq. m
$\Leftrightarrow$side = 33 m';
but 'Rectangle measures 14 yd by 8 yd
$\Rightarrow$area = 112 sq. yd'.

For the 'new mathematics', as for the old, it is necessary for pupils to become accustomed to handling related sets of numbers. First there are sequences; and while these have commonly been treated as a part of algebra their beginning at least is arithmetical. Then there are related sequences such as illustrate functional dependence. Some suggestions will be found in this report for their introduction and use. Finally there are more elaborate two-dimensional patterns such as are found in statistical work. Results of a league competition provide an everyday example, Pascal's triangle a more purely mathematical one.

The suggestions made so far are aimed at preparing the ground for what may come later on. There is also the actual introduction, usually at 'Stage A' level, of some of the new topics. The important thing here is that the work should be relevant to the interests and capabilities of the pupils. It should be capable of being welded into the body of their mathematical knowledge and should be worth teaching for its own sake.

A good example is provided by the binary and other scales of notation. This has topical interest in connection with computers and is of value in helping pupils to understand the place-value system of numeration. It is also quite easy and can be taken at almost any stage. Calculating machines, too, are of interest in themselves and throw light on decimal multiplication and division. Somewhat later the slide rule can serve a similar purpose in increasing the understanding of logarithms.

Suggestions will be found in later sections of this report for early work on statistics, both graphical and computational. Again, choice and chance offer a wide field, too long neglected in elementary work. The forbidding names 'permutations', 'combinations' and 'probability' have obscured the fact that these subjects consist chiefly of counting, and that there are plenty of everyday examples much easier than most of those found in the algebra books.

Examples of sets occur in arithmetic, and the associated language and symbols can be introduced quite early. It is necessary to keep firmly in mind that the symbols express a meaning and should not be manipulated mechanically until much later. The teaching of such things as matrices and groups to young pupils is still in the experimental stage. These subjects can be brought down to 'Stage A' level, but the teaching must be careful and well-prepared, based always on simple examples.

An axiomatic approach to these new subjects is not in general suitable for pupils of school age. The teacher's object should be not to present a ready-made logical system but rather to introduce the pupils gradually to the processes of abstract thinking. It may be that, to professional mathematicians, what is taught in schools is an untidy, illogical, fragmented collection of bits and pieces, mostly physics, resting on no clearly defined basis. But, the nature of children and their learning being what it is, we know that we must proceed from the concrete to the abstract, from the particular to the general.

For instance it is natural and proper for children to begin thinking of a fraction as a broken-off bit of something. As their mathematical concepts are developed, many may appreciate that a fraction is an ordered pair of integers. In this country teachers would work towards this idea. They would never begin with it as a definition: that would be contradicting the whole trend of English mathematical education since the time of the dethronement of Euclid.

The foundations of mathematics are a difficult study and are best explored with a guide who knows when to leave well alone. Mathematics in its evolution has been concerned first with things, then with words and symbols, next with operations and relations, and so with logical structure. The symbols are abstractions and most symbols have more than one interpretation, according to the context. Thus '3' means 'three' (cardinal) and 'third' (ordinal). Some symbols have been casually handled in school teaching (e.g. '=' or '÷' or ':') and reasonable care should be given to defining the sense in which they are used. The 'minus' sign has two distinct meanings* which should be made clear at an early stage. In many other cases a more exact understanding can be developed as the pupil progresses. There also appears gradually the distinction between the exactness of logical consequence and the approximate nature of applications to the physical world. Thus a logical sense is developed and a foundation laid for axiomatic studies later on.

* See p. 8, paragraph numbered 4.

CHAPTER 3

NUMBERS

MATHEMATICS may be said to have two aspects: an outer one when it is applied to the practical affairs of the world, and an inner one when we are considering the relationships between the symbols or elements of the subject. By tradition, most of the arithmetic done in schools has been concerned with practical applications and with the acquiring of skill in computation. This section of the Report deals with work on the numbers themselves, and with the class of problems, of interest to many children from an early age, which serve to focus attention on these number properties and the structure of our number system.

In *The teaching of mathematics in primary schools*, stress was laid on the value of making friends with numbers and noticing the relationships between them. For example, patterns can be discovered by arranging the multiplication table as a square and the laws of numbers such as $a \times b = b \times a$ can be inferred. In the first year of the secondary school such work can provide a new approach to familiar routine calculation and enable a good deal of necessary revision to be done in a novel and interesting way. Work on numbers, accompanied perhaps by a historical approach to number and computation, can provide just that touch of novelty needed to revive interest.

There is a virtually unlimited amount of interesting material available but, if it is to retain its fascination, the teaching of it must be undertaken carefully. This work should be taken in small doses, alongside the usual work with techniques of computation and their application. There are great possibilities for teaching in the puzzle or problem manner. The class is set a problem as a challenge, e.g. '1, $1+3$, $1+3+5$, $1+3+5+7$ etc. are all perfect squares: why is this?' and then the different ways of approach are considered. Watching the pupils at this work will greatly assist the teacher to find out the children's interest and their capacity for dealing with relations between numbers.

Every teacher has his own interests in this field and we suggest that he should select according to his own taste and the response of the class. Problems, paradoxes, glimpses of the way ahead, and digressions into history and etymology all pay dividends by increasing the pupils' interest in the subject. Much useful material will be found in F. C. Boon's *Puzzle papers in arithmetic* and in other works listed at the end of this section. The list that follows is not exhaustive and, while many of these topics may find a place in the course, it is not intended that they should all be included. The reader is referred to Chapter 4 for work on the four rules.

Numerals. The different ways in which numbers have been written in Rome, Greece, Egypt and China, for instance, shed light on the advantages of our own system and serve to remind pupils that there have been civilizations other than our own.

The Roman numerals are important in that they are still in common use. Large numbers were cumbersome to write and the abacus was consequently an almost essential aid to calculating. The Alexandrian system at one time used in Greece had the advantage of writing numbers in a more compact form, but calculation was no more easy. Note that this system of numerals was based on the alphabet, whereas another system used in Egypt as early as 3500

B.C. was based on tallies. It is interesting to compare different systems, noting the number of symbols used, and bringing out the importance attached to place value and zero in our own system.

The growth of our number system. Teachers of mathematics should seize opportunities of developing in their pupils an understanding of our number system. When pupils arrive at secondary schools, they are usually familiar with the natural numbers 1, 2, 3, ..., with zero, and with fractions, but few of them have used directed numbers.

There are several points about which teachers should be clear in their own minds and these are discussed below. It is not intended that all of this should be taught in the early years of the secondary school but mention of the points in question is made in order that teachers should avoid misinforming or confusing their pupils about the subject: 'What the teacher knows and doesn't teach is more important than what he knows and does teach.'

1. There are various types of number in common use in arithmetic; terminology varies slightly and so we explain what we mean by them:

(i) the *natural numbers* are 1, 2, 3, ...;

(ii) *fractions* are numbers in the form $\frac{a}{b}$, where a, b are natural numbers;

(iii) the *integers* are ..., -3, -2, -1, 0, 1, 2, 3, ...;

(iv) *rational numbers* are numbers in the form $\frac{p}{q}$, where p, q are integers and and $q \neq 0$;

(v) *irrational numbers* are numbers, such as $\sqrt{2}$ and π, which cannot be expressed in the form $\frac{p}{q}$, where p, q are integers;

(vi) the phrase *directed numbers* is in common use and may refer to the integers or to the rational numbers.

Then there are other entities in mathematics which obey different sets of laws, for instance complex numbers, matrices, and vectors.

The various types of number have arisen historically as a result of problems in mathematics or in measurement. In the classroom, however, it is not necessary to teach the history of mathematics to see the need for the successive extension of the number system. For instance, if we restrict ourselves to the natural numbers, the operation of subtraction $a-b$ can only be performed if $a>b$. Then if we restrict ourselves to the integers, the operation of division $\frac{p}{q}$ can only be performed if p is divisible by q. Such exceptions to general rules are not pleasing and the wish for general statements leads to extensions of the system of numbers.

2. The same symbol is often used to denote logically distinct numbers, for example '3' may represent the natural number 3, the fraction $\frac{3}{1}$, the directed number $+3$, and even the complex number $3+0i$. There is, however, a (1, 1) correspondence between the corresponding elements of any pair of the ordered sets:

$$\{1, 2, 3, \ldots\},$$
$$\{+1, +2, +3, \ldots\},$$
$$\left\{\frac{1}{1}, \frac{2}{1}, \frac{3}{1}, \ldots\right\}$$

and in practice little difficulty is found in the ambiguous notation.

3. Logically, it is necessary to define the operations of addition and multiplication for the different numbers and then to examine whether the numbers so defined obey the usual 'laws of algebra'.

4. The symbols $+$, $-$, $\times$, $\div$ are used in more than one way. For one thing, these operations are differently defined for the different numbers; this does not cause much misunderstanding, but the distinction between '$-$' as an order to subtract and '$-$' as the prefix of a negative number can and does cause confusion in the minds of some young pupils. For this reason some teachers use, at least for a time, a distinct notation for directed numbers, e.g.:

$$\ldots, (-3), (-2), (-1), 0, (+1), (+2), (+3), \ldots$$

$$\ldots, {}^-3, {}^-2, {}^-1, 0, {}^+1, {}^+2, {}^+3, \ldots$$

$$\ldots, \overset{-}{3}, \overset{-}{2}, \overset{-}{1}, 0, \overset{+}{1}, \overset{+}{2}, \overset{+}{3}, \ldots$$

Having raised what may appear to some readers to be unnecessary complications, we shall try to give some indication of how these matters may be dealt with in the classroom.

1. The existence of different types of numbers will be obvious to pupils as their mathematical education proceeds. Any reluctance to accept a new type of number because of its strangeness may be reduced by some form of visual representation: for fractions, by shaded parts of a whole; for directed numbers by points on a scale; and for irrational numbers such as $\sqrt{2}$ and π, by the length of a line. It also helps to remind the pupils of previous extensions to the number system. The better classes may be given the *reductio ad absurdum* proof that $\sqrt{2}$ is irrational.

2. The use of one numeral to denote different numbers does not usually cause lasting trouble. However, it is worth remarking that a pupil who writes $\frac{5}{6}+\frac{7}{6}=\frac{12}{6}=\frac{2}{1}$ is logically more consistent than one who gives the answer as 2. A common misunderstanding about complex numbers is expressed by saying that i is the square root of the directed number -1. In fact, there is no square root of the directed number -1, but the square root of the complex number $-1+0\mathrm{i}$ is $0+\mathrm{i}$.

3. A consideration of the introduction of addition and multiplication of fractions has been given in Chapter 6. Directed numbers are often considered to be part of an algebra course but some suggestions about their treatment are given below. For a fuller treatment of the subject, see Appendix A2 of the Association's Report, *The teaching of algebra in schools*.

The existence of directed numbers is known to pupils entering secondary schools from the weather forecasts which now give temperatures in Centigrade. It is necessary, however, to make sure that the pupils understand that the origin and the positive direction may be chosen arbitrarily and that the negative is the opposite direction.

When discussing how to add directed numbers, the teacher should be clear in his or her own mind that the law is a matter of definition but will probably not wish to put it quite so bluntly to the class. The definition chosen is not an arbitrary one; rather, the addition and multiplication of directed numbers are chosen to fit in with the addition and multiplication of natural numbers. This is an example of the process of generalisation, which is one of the characteristic features of mathematics. We choose the rules of composition so that

$${}^+a + {}^+b = {}^+(a+b) \qquad \text{(directed numbers)}$$

corresponds to

$$a+b=(a+b) \qquad \text{(natural numbers)}$$

and so that ${}^+a \times {}^+b = {}^+ab$ corresponds to $a \times b = ab$.

In the classroom, addition and subtraction with natural numbers can be represented by movements in opposite directions on a number scale. So addition and subtraction of directed numbers are represented by similar movements. Thus we can interpret $^{+}3 + {}^{-}2$ by some such action as, 'Move to $^{+}3$ on the scale, face the "+" direction and move $^{-}2$ places (i.e. 2 places in the opposite direction). We arrive at $^{+}1$.' So we say that $^{+}3 + {}^{-}2 = {}^{+}1$. Again, for $^{+}3 - {}^{-}2$ we can say, 'Move to $^{+}3$ on the scale, face the "−" direction and move $^{-}2$ places (i.e. 2 places in the opposite direction). We arrive at $^{+}5$.' So we say that $^{+}3 - {}^{-}2 = {}^{+}5$.

An effective way of introducing multiplication of directed numbers is to write up the beginnings of a multiplication table:

	$^{-}3$	$^{-}2$	$^{-}1$	0	$^{+}1$	$^{+}2$	$^{+}3$
$^{+}3$				0	$^{+}3$	$^{+}6$	$^{+}9$
$^{+}2$				0	$^{+}2$	$^{+}4$	$^{+}6$
$^{+}1$				0	$^{+}1$	$^{+}2$	$^{+}3$
0				0	0	0	0
$^{-}1$							
$^{-}2$							
$^{-}3$							

The class usually suggests that the product of a positive and a negative number should be negative, and then the sequences in the rows and columns bring most of the pupils to see that the product of two negative numbers should be taken to be positive. To some a more convincing method is to choose some application which is familiar to the class and which makes sense with negative numbers, for example, for constant speed,

$$\text{distance} = \text{speed} \times \text{time};$$

or, for a clock gaining or losing at a constant rate,

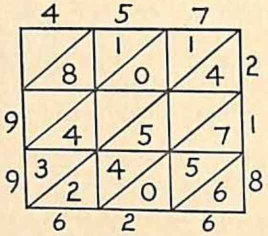
$$\text{gain} = \text{rate} \times \text{time}.$$

The rules of multiplication of directed numbers can then be chosen to fit in with the physical interpretation.

It should be emphasised that operations on directed numbers should only be introduced *when the need for them arises*. This means that multiplication and division may well be considered much later than addition and subtraction unless, of course, a pupil raises the question of how these operations are carried out. Time spent on clarifying the pupils' ideas on the number system is time well spent. It will ease the treatment later on of surds and complex numbers; but above all it will help to show the ordinary pupil that the rules of mathematics are reasonable, and not arbitrary.

Methods of calculation. Our present methods of long multiplication and division were developed long after the invention of our number system. Their use depends on the ability to add and subtract, and on a knowledge of multiplication tables up to 10 × 10. These were probably not commonly memorised in this country, even by educated people, until the sixteenth century. Consequently a number of other methods have been employed.

The abacus dispenses with the need for a knowledge of any number relationships and, in the hands of a skilled calculator is as fast as our written methods. The 'grating' method of multiplication was used in many countries, including India and China. It is of interest because of the neat diagonal device for moving on the 'tens' figure in each of the partial products. Here it is shown for 457 × 218 = 99 626.



Before the adoption of our present method of long division, which was not in general use in England until the beginning of the eighteenth century, the 'galley' or 'scratch' method was common. 384 is divided by 13 below. The finished calculation would appear as in (vi), but the steps are shown with explanations below each stage to show what has just been done.

$$\begin{array}{ccc|c} 3 & 8 & 4 & \\ 1 & 3 & & \end{array}$$

(i)

$$\begin{array}{ccc|c} 1 & & & \\ \cancel{3} & 8 & 4 & 2 \\ \cancel{1} & 3 & & \end{array}$$

(ii)
Subtract 10 × 20.

$$\begin{array}{ccc|c} 1 & 2 & & \\ \cancel{3} & \cancel{8} & 4 & 2 \\ \cancel{1} & \cancel{3} & & \end{array}$$

(iii)
Subtract 3 × 20.

$$\begin{array}{ccc|c} 1 & 2 & & \\ \cancel{3} & \cancel{8} & 4 & 2 \\ \cancel{1} & \cancel{3} & 3 & \\ & 1 & & \end{array}$$

(iv)
Write down divisor again.

$$\begin{array}{ccc|c} & 3 & & \\ \cancel{1} & \cancel{2} & & \\ \cancel{3} & \cancel{8} & 4 & 2\ 9 \\ \cancel{1} & \cancel{3} & 3 & \\ & \cancel{1} & & \end{array}$$

(v)
Subtract 10 × 9.

$$\begin{array}{ccc|c} & \cancel{3} & & \\ \cancel{1} & \cancel{2} & 7 & \\ \cancel{3} & \cancel{8} & \cancel{4} & 2\ 9 \\ \cancel{1} & \cancel{3} & \cancel{3} & \\ & \cancel{1} & & \end{array}$$

(vi)
Subtract 3 × 9.

The quotient is 29 and the remainder is 7.

The inventions of decimal and common fractions, and of logarithms, deserve some attention. Books on the history of mathematics give details of these and many other aspects of the history of calculating.

Some teachers use calculating machines to give a fresh approach to number operations and the decimal notation.

Scales of notation. A study of scales of notation, besides being of interest to many pupils, provides an opportunity to develop in them a greater insight into the way in which we usually write our numbers; this is especially important when they are starting on decimals. By the secondary stage, pupils are familiar with numbers to the base ten and they have a store of associations with written numbers up to about 100. Thus 39 is an odd number, it is equal to 3 × 13, and so on. These associations will not be unduly upset by experiments in using figures to write numbers to other bases. Thus 42 in decimal notation is 4 tens and 2, or 6 × 7, and 42 pence is 3s. 6d. But the digits 4 and 2 in 42 when the base is *five* indicate 4 fives and 2, i.e. twenty-two in decimal notation. Again, in the table of threes, for example,

	Scale of ten	*Scale of three*
1 three =	3	10
2 threes =	6	20
3 threes =	9	100
4 threes =	12	110
5 threes =	15	120
6 threes =	18	200
7 threes =	21	210
8 threes =	24	220
9 threes =	27	1000

It will be noted that the scale of three involves only three digits, 0, 1 and 2.

The use of ten fingers for the purpose of tallying led naturally to the use of a decimal system of notation. Had the human race generally used the twelve finger joints of one hand, thumb excluded, for the same purpose, we should have the duodecimal system instead. The history of mathematics provides other examples, but the advent of electronic calculating machines has brought the scale of two into prominence. Since only the symbols 1 and 0 are needed to represent numbers, machines can be made to represent a number by a set of electric currents which either do or do not flow. The scale of two has other well-known applications. In kitchen scales, the commonly used weights are $\frac{1}{4}$, $\frac{1}{2}$, 1, 2, 4, 8 oz and 1, 2 lb. (It is an interesting digression to consider what weights are needed if they may be placed on both sides of the scales.) 'Peasant' multiplication, age-telling cards and the game of Nim all make use of the scale of two.

Prime numbers and prime factors. The expressing of numbers in prime factors has an interest of its own, besides being a useful preliminary to factor work in algebra. It leads to the discovery of the sequence of prime numbers and their curiously irregular distribution, and it also provides occasion for the introduction of indices and allows the index laws to be discovered through simple examples. Work on prime factors should not involve large numbers but one of its aims should be to help the pupils to become familiar with the factors of the smaller numbers, say less than 100, together with squares, cubes and numbers such as 112 which frequently occur in calculations. Tests of divisibility arise naturally from this work and can be discussed in the course of it; treated this way they will be of much greater interest than if they are taught formally. Square roots, and possibly cube roots, can be found by prime factors at this stage and, while an introduction to fractional indices may not be appropriate so early in the course, it is likely that some pupils will ask what happens when the indices are not exactly divisible by two or three respectively, e.g.:

$$\sqrt{5^3}, \quad \sqrt[3]{2^4},$$

and a seed may be sown which will prepare them for later work.

Other topics. Easy sequences, the integers, squares, cubes and triangular numbers all have interesting applications and are readily linked with algebra through the finding of a formula for the *n*th term. A little work on recurring decimals is usually called for by the pupils' questions. The concept of number needs to be extended for later work, although this may be more suitable for the algebraic side of the mathematics course.

The L.C.M. and H.C.F. are only worth doing if they are properly understood, and of the two, the Least Common Multiple is the more important. It gives precision to the idea of the common denominator when fractions are added, and may well be introduced as the Least Common Denominator. This is particularly useful in algebra when the denominators have common factors. First the teacher must make sure that all the pupils understand the meaning of the word 'multiple'. From there, the next step is to find multiples which are common (another word which will not be understood by everyone) to two or more numbers, and then the pupils may find some least common multiples. As with most parts of arithmetic, the emphasis in teaching needs to be on the underlying ideas of the work; the way in which the work is to be set out will vary from child to child, with the teacher or other pupils suggesting improvements. The Highest Common Factor, if undertaken, can readily be treated in a similar manner—not immediately afterwards, but only when the method of finding the L.C.M. has been mastered.

For those teachers who wish to learn more about such topics as have been mentioned in this chapter, a list of books is included.

Ball, W. W. R., and Coxeter, H. S. M., *Mathematical recreations and essays.* (Macmillan)
Boon, F. C., *A companion to school mathematics.* (Longmans)
Boon, F. C., *Puzzle papers in arithmetic.* (Bell)
Bowman, M. E., *Romance in arithmetic.* (U.L.P.)
Dantzig, T., *Number, the language of science.* (Allen and Unwin)
Heafford, P., *Mathematics for fun.* (Hutchinson)
Hooper, A., *Makers of mathematics.* (Faber)
Kraitchik, M., *Mathematical recreations.* (Allen and Unwin)
Land, F., *The language of mathematics.* (Murray)
McKay, H., *Odd numbers.* (O.P.)
Reid, C., *From zero to infinity: What makes numbers interesting.* (Routledge & Kegan Paul)
Sanford, V., *A short history of mathematics.* (Harrap)
Shackle, G. L. S., *Mathematics by the fireside.* (O.P.)
Smith, D. E., *History of mathematics:* Vol. I, *General survey of the history of elementary mathematics.*
Vol. II, *Special topics of elementary mathematics.* (Dover)
Smith, D. E., *Source book in mathematics*, 2 vols. (Dover)
Smith, D. E., *Number stories of long ago.* (Ginn)
Smeltzer, D., *Man and number.* (A. & C. Black)
Turnbull, H. W., *The great mathematicians.* (Methuen) (Also E.U.P. paperback)

CHAPTER 4

SOME REMARKS ON THE FIRST FOUR RULES

AT first a child's main interest in the first four rules lies in what he can do with them. He is chiefly concerned with getting the answer. By the time he has reached the secondary school it ought to be possible to divert some interest to the operations themselves and this can be valuable preparation for later work. We might begin by noticing that, in the case of addition and multiplication, the order in which the numbers are taken does not influence the answer. Then we ought to recognise that the four operations are not independent, but subtraction is another aspect of addition and division another aspect of multiplication. Questions such as the following will lead up to these points: What number must be added to 72 to give 95? By what number must 8 be multiplied to give 48? A number is divided by 13 and the answer is 15; what is the number? Practice in the correct reading and writing of particular number statements can be brought in here:

$3+5=8,$	'3 plus 5 equals 8';
$8-3=5,$	'8 minus 3 equals 5';
$5\times 3=15,$	'5 multiplied by 3 equals 15';
$\left.\begin{array}{r} 15\div 3=5, \\ \text{or } \dfrac{15}{3}=5, \end{array}\right\}$	'15 divided by 3 equals 5'.

Thus we reach some understanding of two fundamental truths:

(i) Subtraction answers the question 'What must be added to A to give B?'
(ii) Division answers the question 'By what must A be multiplied to give B?'

This is useful preparation for later work. For example,

(i) If $3x=11$, then $x=\frac{11}{3}$;
(ii) If $x+3=8$, then $x=8-3$;
(iii) From $(-5)+(+11)=(+6)$ we can deduce
$(+6)-(-5)=(+11)$
and hence the rule that $-(-5)$ is equivalent to $+(+5)$.

When we come to velocity, the statement

'(the velocity of an aircraft relative to the earth)
=(velocity of the aircraft relative to the air)
+(velocity of the air relative to the earth)'

may be re-written

'(the velocity of an aircraft relative to the air)
=(velocity of the aircraft relative to the earth)
–(velocity of the air relative to the earth)'

and this leads to the general rule for finding the velocity of A relative to B.

It is interesting to note that the addition aspect of subtraction is the basis of what is probably the commonest use of subtraction in everyday life, the reckoning of change. In finding what change to give from a 10s. note after making a purchase costing 2s. 8½d., the process always is '2s. 8½d. and 3½d. makes 3s., and 7s. makes 10s.'

The fundamental processes are bound to come up for review from time to time in the early stages at the secondary school. In general it is *not* advisable to try to alter methods of computation which are thoroughly known. This may mean that several methods, particularly in subtraction and multiplication, are in use in a class, and occasionally the teacher has to remember to consider separately the needs of the children using alternative methods. This inconvenience is preferable to compelling children to unlearn something that has already become mechanical. Such consideration of fundamental techniques as is necessary should aim at clarifying principles. For example the fairly common experience of finding a pupil dividing by 1000 by long division should be made an excuse for a further discussion on place value. Other examples bearing on the same principle are multiplications like 3742 × 1003 where many pupils, left to themselves, would write down two rows of four noughts.

The method of long division is recommended when it is desired to find a remainder. Here again it may be worthwhile looking at the mechanical process in a different way in order to achieve greater understanding. For example

$$
\begin{array}{rrcl}
 & 14 & & \\
13\,) & \overline{186} & & \\
 & \underline{13} & \ldots\ldots\ldots & 130 = 10 \times 13 \\
 & 56 & & \\
 & \underline{52} & \ldots\ldots\ldots & 52 = 4 \times 13 \\
 & 4 & \ldots\ldots\ldots & \underline{4 = 4} \\
 & & & 186 = 14 \times 13 + 4
\end{array}
$$

Division by factors can be useful when only the quotient is required.

The aim throughout this early work should be the progressive introduction of mathematical notation, and usage. For example, the statement of an answer to a division in the form of quotient and remainder should give way, in suitable cases, to an answer expressed as a mixed number (with or without decimals). Moreover, the pupil should gradually be weaned from his natural habit of working out each stage of a calculation before proceeding to the next. The aim to set before him is the formulation of the answer as a mathematical statement before the calculation is begun, and it can be pointed out that by doing so the work can often be shortened.

CHAPTER 5

COMPOUND UNITS

IN the application of numbers to measurement, the basic idea is to define a unit and count how many of them are needed to make up the quantity to be measured. When the need for greater precision is felt there are two main lines of advance:

1. by the use of a set of related units such as yards, feet and inches so that we can continue to count in integers;
2. by extending the concept of number to fractions and decimals and continuing to use a single unit of measure such as the yard.

No doubt the first method seemed the more natural—it was certainly the first to be tried—but as calculation became more prominent the limitations of this method became more obvious. In 1897 the weight of an Admiral's barge was given in a manual on seamanship as 1 ton 7 cwt 2 qtr 26 lb. Compare this with the following particulars of a motor car taken from a recent copy of a motoring journal: price £1207 1s. 1d., capacity 1485 c.c., weight 21¼ cwt, maximum speed 82·4 m.p.h. There has evidently been a swing towards the second method. One advantage of using a single unit is that the degree of accuracy is more obvious since it is merely necessary to note the number of significant figures. In ordinary everyday usage the same preference for a few units can be seen. A man's weight and height are given as 12 st 7 lb and 5 ft 10½ in. rather than as 1 cwt 4 st 7 lb and 1 yd 2 ft 10½ in., 5½ yd of cloth is more usual than 5 yd 1 ft 6 in., and a price is just as likely to be quoted as 37s. 6d. as in the form £1 17s. 6d. It is fair to say that teachers and examiners have been slow to follow this development and calculations like finding the cost of 43 things at £7 13s. 9½d. each have tended to persist in schools longer than anywhere else. No doubt such exercises provide excellent tests of speed and accuracy, but they delay the advance to matters of greater mathematical interest. The useful purposes served by these applications of arithmetic in the school course are to develop skills that may be needed when schooldays are over, and to deepen present interest by relating the work of the classroom to the real world outside. It is therefore particularly important here that exercises should be realistic and as far as possible based on material which is meaningful to children. The number of units used in a calculation should not exceed two, except in the case of money, and even there we need not always use three.

Applications to money, weights and measures begin with experience of the units, and the relationships between them. Money is particularly convenient as an introduction since experience can be gained by actually handling coins, although by this very fact it is limited to special fractions of a sum of money. On the other hand measures can be subdivided at will and this is well illustrated on the ruler where the child can find alternative sub-divisions of the inch. Thus linear measurement should be introduced early. Weights can also be handled but the idea of weight is not such an easy one. It is easy to underestimate the value of such experience in preparation for the learning of tables and conversion from one unit to another. Some teachers hold that this later work is simply the systematisation of previous experiences. No doubt in the case of the basic measures this experience will have been gained before the secondary school stage, but we may be well advised to remember its importance when we are undertaking remedial work or introducing new units. For example, it is very well worth while for the pupils to draw inch squares and actually handle inch cubes. If we seldom work with more than two units, conversion from one unit to another can be informal and the special setting-out of the old 'reduction' sums is not needed at all. Such parts of the tables as still need learning can be dealt with piecemeal as the need arises, and some of the traditional units can be omitted or mentioned only for their historical interest. In length measure, furlongs and chains are worth some attention, but the 'rod, pole or perch' is now obsolete. In square measure, the acre is the only special unit needed: its connection with furlongs and chains is worth noting. In weight measure, tons and hundredweights must be known, but quarters are not worth more than a mention. It should perhaps be more widely known that in America a hundredweight is 100 lb and a ton 2000 lb (but the 'metric ton' is 1000 kilogrammes and thus nearer to the English ton).

If pupils are to become thoroughly conversant with the metric system, it is highly desirable that they should start using it early. In this connection, scale drawing and graphical work using both the metric measures and inches with tenths, estimating a second decimal place when needed, can be invaluable. It is helpful, too, to use centimetre graph rulings as well as inch rulings.

The prefixes hecto-, deca-, deci-, and mega- should be introduced for completeness, and because of their occasional use, but regular practice should be confined to the commoner units. (See Appendix.)

A class is always interested to learn that the metric system originated in France after the French Revolution in a scientific attempt at reform, the French system before that having been as chaotic as our own. It never fails to amuse the pupils to hear that the metre, which was intended to be one forty-millionth of the circumference of the Earth, had to be otherwise defined in the end owing to an error in the original measurement. (See Appendix.)

The surveyor, the seaman, the silversmith, the chemist, the corn chandler and many others have their own units for their own purposes and some of these are worth mentioning as curiosities. Some too have interesting histories: thus the pennyweight was the weight of a silver penny, so naturally there were 240 pennyweights in a pound. But the days have gone by when schoolchildren were expected to know and use a wide variety of special tables.

Compound multiplication. Addition and subtraction of compound quantities will give little trouble beyond the correction of occasional lapses due to confusing the decimal scale with other scales. Compound multiplication may need more attention, but not necessarily all at once nor at the beginning of the course. We recommend that no single 'method' should be unduly emphasised but the habit of considering alternatives should be formed right from the start. Three general ways of organising compound multiplication can be suggested:

1. *Conversion to a single unit*. If the compound quantity is expressed in terms of the smallest unit, the calculation is reduced to the familiar one of multiplying integers. This is a thoroughly sensible procedure and must never be described as wrong. No doubt we shall take every suitable opportunity of pointing out the advantage of converting to a larger unit and using fractions. Some drill in the fractional parts of the shilling and pound may be advisable, otherwise the use of fractions may not be readily accepted. Later on, when logarithms are available, the advantage of approximate decimalisation becomes clear, but in the case of multiplication the use of this method is limited by the gross inaccuracy of the answer due to the multiplication of the error.

2. *Practice*. It has been customary to introduce this method at a compara-

tively late stage with a formal setting out that makes it one of the traditional 'sums' of school arithmetic. The method is commonly applied to questions involving money but even if decimal coinage is introduced we still recommend that the idea should be introduced informally at an early stage. For instance, a child who is accustomed to using kitchen scales knows that to weigh 2 lb 13 oz he needs the following weights: 2 lb, 8 oz, 4 oz, 1 oz, and so he might well set down the working of 2 lb 13 oz × 23 as follows:

	lb	oz
2 lb × 23	46	0
8 oz × 23	11	8
4 oz × 23	5	12
1 oz × 23	1	7
	64	11

A child who is still thinking in terms of coins might work a multiplication like £1 15s. 9d. × 23 as follows:

	£	s	d
23 pound notes	23	0	0
23 ten shilling notes	11	10	0
23 halfcrowns	2	17	6
23 ,,	2	17	6
23 sixpences		11	6
23 threepences		5	9
	41	2	3

No doubt 23 threepences would be expressed in shillings and pence by reckoning 4 threepences to the shilling first, but the simple connection with the previous line might be pointed out as a check, and can be adopted as a shorter method of calculation later on.

Then the widespread use of ready reckoners can be pressed into service. The class can be set to work in groups to make a ready reckoner for a milk roundsman when the price is, say, 8½d. a pint. The costs of 1 pint, 2 pints, 3 pints, ..., 9 pints; 10 pints; 20 pints, 30 pints, ..., are worked out and brought together. It is vital that these results should be correct and methods of checking them can be discussed. Thus the differences between the costs of 1, 2, 3, ... pints should be constant, and so should the differences between the costs of 10, 20, 30, ... pints. Also, and more usefully from our point of view, the cost of 20 pints should be 10 times the cost of 2 pints, and so on. Thus prominence is given to the relationship between different calculations. Or we might start with a price of 6d. per pint, work out the ready reckoner and then say, 'The price rises to 7d., have we to start our calculations all over again?' Thus we introduce the idea of calculating, say, 7d. × 23 as 6d. × 23 + 1d. × 23 Even with the simplest calculations it is worth showing the use of this idea as one possibility. For example, the cost of 27 books at £1 10s. each can be found as follows:

	£	s	d
Cost at £1	27	0	0
Cost at 10s.	13	10	0
Cost at £1 10s.	40	10	0

Often the idea comes as a pleasant surprise and is taken up with enthusiasm, especially as there is scope for ingenuity in choosing aliquot parts. On the other hand, less able pupils sometimes find this choice a difficulty and it may be advisable to discuss the method of calculation in a large number of cases which are not fully worked out.

3. *Long multiplication*. It is doubtful whether there is much point in teaching any particular formalised setting out, and there is no case at all for teaching one as *the* method of multiplication. This attributes more importance to the position of figures on a page than to principles and spreads the view that mathematics is a matter of knowing the right rules rather than of applying commonsense.

Compound division. This can be a complicated process and is now little used, but two points may be noted.

1. It is often convenient to reduce the dividend to a single unit, but this can never be a rigid rule.
2. If the divisor is a small number, or easily factorised, short division will often be preferable to long, whereas if it is a larger or more awkward number, then long division or logarithms and a single unit should be used.

For instance, if a farmer floods his field for skating with 931 000 gallons of water and is charged £10 6s. 6d. by the Water Board, the cost per 1000 gallons is

$$\frac{2478}{931}\text{ pence} \simeq 2{\cdot}66 \text{ pence};$$

but if 4 boys share the cost of a camp, which amounted to £16 12s. 8d., then each will pay

$$\frac{\pounds 16\ 12\text{s. } 8\text{d.}}{4} = \pounds 4\ 3\text{s. } 2\text{d.}$$

Pupils should be encouraged, from their early days, to discuss the possible methods and exercise their own choice in deciding which to use.

Decimalisation of quantities in compound units. Some practice in decimalisation is necessary both in readiness for division and also for the use of logarithms. There are special methods for rapidly expressing money as a decimal of £1 to a limited number of places, but it is better to rely on the direct method which is of general application.

Express £2 3s. 5d. *as a decimal of* £1.

$$\begin{array}{r l} 12\,)\,\underline{5{\cdot}00} & \text{d.}\\ 20\,)\,\underline{3{\cdot}41666} & \text{s.}\\ 2{\cdot}17083 & \pounds \end{array}$$

∴ £2 3s. 5d. = £2·1708, to 4 decimal places.

Express 3 *tons* 550 *lb in tons to* 4 *sig. fig.*

$$3\frac{55}{224} = 3{\cdot}2455 \ldots \text{ by long division.}$$

$$\therefore\ 3 \text{ tons } 550 \text{ lb} = 3{\cdot}246 \text{ tons to 4 sig. fig.}$$

The converse process is also needed.

Write £2·171 *in £ s. d. to the nearest penny.*

$$\begin{array}{rl} \pounds 2{\cdot}171 & \\ 20 & \\ \hline 3{\cdot}42 & \text{s.} \\ 12 & \\ \hline 5{\cdot}04 & \text{d.} \end{array}$$

∴ £2·171 = £2 3s. 5d. to the nearest penny.

The above examples show the methods of calculation advocated but it is not suggested that any particular way of setting down the working is necessarily the best.

INSTITUTE OF EDUCATI[illegible] WOMEN ★ CALCUTTA-27 ★
Dept. of Extension
SERVICE.

CHAPTER 6

FRACTIONS AND DECIMALS

FRACTIONS have grown out of a need felt increasingly through the ages to measure quantities more accurately. At first this need was met by introducing smaller units in order to avoid the use of fractions. This is understandable when it is remembered that even computation with integers used to be regarded as a special skill. Nevertheless, fractions of one sort and another have been used at least since the time of the Ahmes Papyrus (*c.* 1550 B.C.). There was a tendency among the ancients to use only unit fractions, i.e. those with a unit numerator. This led to expressions (in modern notation) such as

$$\frac{1}{42}+\frac{1}{86}+\frac{1}{129}+\frac{1}{301}$$

for the fraction $\frac{2}{43}$. Another system of fractions used from early times was the sexagesimal system which we still use for fractions of a degree, e.g. 21° 35′ 46″, but further places were used in the Middle Ages. To distinguish between fractions in the form $\frac{a}{b}$ and sexagesimal fractions, the former were called 'common fractions', *fractiones vulgares* in Latin, and hence 'vulgar fractions'.

Decimal fractions may be said to have been invented in the sixteenth century A.D. by Christoff Rudolff but his work does not appear to have been appreciated. Fifty-five years later, in 1585, Stevin published an account of how decimal fractions could be used for all business purposes.

Common and decimal fractions serve the same purpose and are interchangeable in that every common fraction may be expressed as a terminating or recurring decimal, and every terminating or recurring decimal can be expressed as a common fraction. From this it might appear that one system is redundant but each has its uses: *common fractions* represent rational numbers simply, they are well adapted to exact calculation, and they link well with the idea of ratio; *decimal fractions* enable comparisons between fractions to be made at sight and so are appropriate for recording measurements, and they have the advantage that the four rules are basically the same as for integers.

It cannot be laid down whether common or decimal fractions should be introduced first. At the time of writing, infants have some experience of simple common fractions and they are more likely to wish to record such fractions rather than those involving tenths. Their use of any mathematical notation should grow out of a wish to express their experience in written form and, with the introduction of decimal coinage, there will be a tendency for the decimal notation to be used earlier than is general at present. Neither notation should be taught to the exclusion of the other but rather the study of one type of fraction should enrich the understanding of the other.

Fractions

The comparison of simple fractions is a useful exercise. In this and in addition of fractions, stress should be laid on the principle involved in bringing fractions to a common denominator, viz. that a fraction is unaltered if its numerator and denominator are multiplied by the same number. At first fractions should be written with the same denominator separately before the fractions are combined. In algebra, too, pupils starting to add fractions will find the process easier to understand if they are accustomed to writing the denominators separately. In general, mixed numbers should not be reduced to improper fractions in addition and subtraction.

Computation with fractions should be simple: it is not easy to find the need in mathematics even for the addition of, say, $\frac{2}{5}$ and $\frac{5}{8}$. It is harder still to find the need for the addition or subtraction of three or four fractions until the calculus is reached and definite integrals of polynomials have to be evaluated.

The rules for manipulating fractions are not merely a logical extension of those for integers. The teacher should be aware of this as it may affect his attitude and method of approach. From the logical point of view, a fraction is a pair of numbers and rules for the manipulation of such pairs could be laid down in a variety of ways. One particular way is chosen because it fits in with practical requirements and this is a justification for the usual classroom approach in which the rules are built up from practical examples.

A first consideration of the four rules for fractions should grow out of practical and, if possible, concrete situations. The primitive ideas of addition and subtraction as 'putting together' and 'taking away' are easily carried over into situations where quantities are expressed as fractions. But this is not the case with multiplication and division by fractions. The child who is only used to multiplication by integers thinks of the process as one which 'makes bigger' and multiplication by a fraction may appear as a contradiction in terms. This means that, although there is logically a need to define addition of and multiplication by fractions, it is much easier to give a psychologically satisfying explanation of the former process.

In dealing with integers, we proceed from a statement such as '2 oranges + 3 oranges = 5 oranges' to the abstract '2 + 3 = 5', and a similar method is necessary for fractions. To show the meaning of a fraction $\frac{3}{4}$, we may divide a length, area or sum of money into four equal parts and take three of them. It is easy to see that if we divide into eight equal parts and take six of them the result is the same. From this we deduce the abstract statement that $\frac{3}{4}=\frac{6}{8}$. From the fact that, if three of the eight parts are added to two more, the result is five such parts we conclude that $\frac{3}{8}+\frac{2}{8}=\frac{5}{8}$. In such a way we build up the rule for addition of fractions.

The need for multiplication by a fraction may be suggested in a number of ways. In each of these there will arise a stage where multiplication by a fraction is defined. Two suggestions are given below.

Areas. If a class is familiar with areas of rectangles whose lengths and breadths are integral numbers of units, they may be asked to find the areas of rectangles $\frac{1}{3}$ in. by 2 in., 2 in. by $\frac{1}{3}$ in., $\frac{1}{2}$ in. by $\frac{1}{3}$ in.

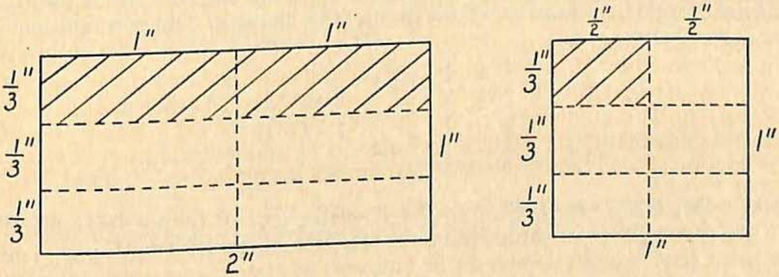


FIG. 6.1

Figures such as those above will quickly lead them to the answers. Other rectangles can be treated similarly. When lengths and breadths are integral numbers of units, the area is found by *multiplication* and so the class may be led to enlarge their concepts of multiplication to include multiplication by fractions. An advantage of this method is that it can be illustrated visually in the classroom.

Commercial examples. This approach has been used widely as a means of introducing the rules for multiplying and dividing by fractions. The approach depends on proportion. The class consider some such illustration as the buying of cloth. It is evident that the cost of 5 yards is 5 times the cost of 1 yard, i.e. the cost of one yard multiplied by 5. For the cost of $\frac{3}{4}$ yard we divide the cost of one yard by 4 and multiply by 3, i.e. we take $\frac{3}{4}$ of the cost of one yard. It is convenient to call this 'multiplication' as in the other case. Thus multiplication by $\frac{3}{4}$ is identified with taking '$\frac{3}{4}$ of' the cost of one yard.

The fraction $\frac{15}{5}$ (fifteen fifths) may be simplified to 3 (three wholes) and a comparison with '$15 \div 5 = 3$' leads, with similar examples, to a realisation that the fraction bar may be interpreted in two ways: as part of the notation that we use for writing fractions, and as an instruction to divide. In general it is better to use the fractional notation rather than the division sign. Logically, the rational number $\frac{15}{5}$ or $\frac{3}{1}$ is distinct from the integer 3 because the rational numbers are *pairs* of integers. In practice, however, the distinction is not important since a fraction such as $\frac{3}{4}$ is represented by dividing something into 4 equal parts and taking 3 of them.

Pupils entering the grammar school are usually familiar with the rule for division, 'To divide by a fraction, turn it upside down and multiply,' but they can seldom explain why it works. Perhaps the most direct method is to say, for example, that

$$\frac{\frac{3}{5}}{\frac{2}{7}} = \frac{\frac{3}{5} \times \frac{7}{2}}{\frac{2}{7} \times \frac{7}{2}} = \frac{3}{5} \times \frac{7}{2}.$$

Alternatively, both fractions may be expressed with the same denominator, e.g.

$$\frac{\frac{3}{5}}{\frac{2}{7}} = \frac{\frac{21}{35}}{\frac{10}{35}} = \frac{21}{10}.$$

Another explanation, which is favoured by many teachers, is to multiply numerator and denominator of the quotient by the least common multiple of the denominators, e.g.

$$\frac{\frac{3}{5}}{\frac{2}{7}} = \frac{\frac{3}{5} \times 7 \times 5}{\frac{2}{7} \times 7 \times 5} = \frac{3 \times 7}{2 \times 5} = \frac{21}{10}.$$

This looks longer on paper but with practice the two middle steps are not written down and it is easy to simplify fractions by this process, e.g.:

$$\frac{2\frac{1}{2}}{3\frac{1}{2}} = \frac{5}{7} \quad \text{and} \quad \frac{\frac{1}{a} + \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}} = \frac{b + a}{b - a}.$$

There is a difference of opinion among teachers as to the wisdom of allowing the numerator and denominator to be scratched through when 'cancelling' is done. It is recommended, however, that crossings out should be kept to the minimum.

$$\frac{2}{14}=\frac{1}{7}$$

is clearer than

$$\frac{\overset{1}{\cancel{2}}}{\underset{7}{\cancel{14}}}=\frac{1}{7}.$$

and good results can be obtained in more complicated examples without any crossing out, e.g.

$$\begin{aligned}\frac{7}{13}\times\frac{9}{14}\times\frac{26}{12}&=\frac{1}{1}\times\frac{3}{2}\times\frac{2}{4},\\ &=\frac{3}{4}.\end{aligned}$$

For the first step of this method the lines of the fractions and the multiplication signs are written first and the figures then inserted in the appropriate spaces. When all the spaces are used up another step may be needed if there is further cancelling to be done, but this can often be disposed of mentally as in the above example.

If crossing out is permitted, plenty of space is required above and below, and the work must be kept neat and legible. Mistakes in accuracy are frequently caused by small or carelessly written figures during this operation. Checking is easier if different types of stroke are used for the different stages of the 'cancellation', e.g.

$$\frac{\overset{1}{\cancel{7}}}{\underset{1}{\cancel{13}}}\;\frac{\overset{3}{\cancel{9}}}{\underset{2}{\cancel{14}}}\;\frac{\overset{2}{\cancel{26}}}{\underset{4}{\cancel{12}}}=\frac{3}{4}.$$

In no event should pupils be allowed to cross out a number by drawing a separate line through each digit. Difficulties of printing lead to numbers being crossed out in this way in the examples shown in some books.

In connection with fractions the pupil is brought up against the 'associative' and 'distributive' laws of algebra. Thus

$$3\times2\tfrac{1}{5}=6\tfrac{3}{5},\quad\text{but}\quad 3\times2\times\frac{1}{5}=6\times\frac{1}{5},\quad\text{not}\quad 6\times\frac{3}{5};$$

$$\text{and}\quad\frac{50\times3}{60}=\frac{5\times3}{6},\quad\text{but}\quad\frac{50+3}{60}\quad\text{is not equal to}\quad\frac{5\times3}{6}.$$

'Associative' and 'distributive' are long and repellant words which need not be used in the classroom; if any names are needed for these laws they may be referred to as the 'any-order rule' and the 'do-it-to-each rule'. Wrong ideas in this kind of work persist to a much later stage: for example, when dealing with area of a triangle, pupils are sometimes found to be uncertain whether *half base × height* is the same as $\frac{\text{base}\times\text{height}}{2}$. The habit of 'cancelling' sometimes degenerates into the crossing out of any two numbers that are the same, e.g.

$$\frac{a+x}{a}=x, \quad \text{or} \quad (1+x).$$

Such errors occur in the work of quite advanced pupils and teachers must always be prepared to refer back to such simple illustrations as

$$\frac{20\times 30}{10}=2\times 30 \quad \text{and} \quad \frac{20+30}{10}=2+3.$$

Geometrical illustrations may also be used. Thus we can double the perimeter of a rectangle by doubling both the length and the breadth; but to double the area we double only one of them.

Decimals

Decimal fractions are being used more and more widely in the world. They lend themselves readily to computation with tables, slide rules, calculating machines and computers, and this is probably the main reason for their advancement as against common fractions. For similar reasons, decimal coinage and the metric system of measurement are also used far more extensively than at the time of the 1932 Report. Children are likely to have acquired some knowledge of the decimal system of notation at quite an early age, and it is on the basis of such knowledge that a formal treatment of the use and manipulation of decimals can be built.

There is no need to link the introduction of decimals with the metric system. For instance, an exercise in geometry might lead to the measurement of a length in inches and the convenience of the notation can be pointed out. If the class are able to estimate a length to the nearest hundredth of an inch, the notation can be extended to two decimal places: there is no point in working with more than two decimal places in the early stages. A natural use for the second decimal place occurs in finding areas by counting squares on a $\frac{1}{10}$ in. grid.

Place value is the aspect of the decimal notation which makes it so easily used in machine methods of computation but, although place value is so important, it tends to be forgotten as new processes and ideas are introduced, and it is essential that attention be drawn to it from time to time. Some teachers find that calculating machines provide an effective way of doing this; others introduce the binary and other scales for the same reason. At some stage numbers should be written in column form (hundreds, tens, units) and operated on, e.g. multiplied by 10 and the figures moved into the appropriate columns. Then multiplication by 100 can be performed and the consequent movement of figures noted. Extension of the columns to the left (i.e. thousands, ten-thousands, etc.) will be necessary and should help with understanding. This can be followed by examples of division by 10 and 100, with the resulting movement of figures towards the right. Finally, after enough examples of multiplication and division by powers of 10, the pupil should be able to form the rules for himself and the extension of the columns to the right (tenths, hundredths) presents little difficulty.

Addition and subtraction are easy; the only idea to be stressed is that, with the aid of the decimal point, the place value of the digits must be meticulously preserved.

Addition

$$\begin{array}{r} 13{\cdot}5 \\ 2{\cdot}48 \\ 0{\cdot}07 \\ 1{\cdot}1 \\ \hline 17{\cdot}15 \\ \hline \end{array}$$

Subtraction

$$\begin{array}{r} 82{\cdot}5 \\ 4{\cdot}79 \\ \hline 77{\cdot}71 \\ \hline \end{array}$$

The close connection between fractions and decimals can be demonstrated by the conversion of one into the other, e.g.

$$\frac{1}{4}=1\div 4, \qquad 4\,\overline{)\,1{\cdot}00}\atop \quad 0{\cdot}25$$

and

$$0{\cdot}075=\frac{75}{1000}=\frac{3}{40}.$$

But many fractions are not convertible into exact decimal forms and this should lead to profitable discussion on the number of decimal places required or usefully appropriate. In this connection (or, for example, with ratio problems) conversion of fractions such as $\frac{2}{3}$, $\frac{7}{11}$, $\frac{3}{7}$ to decimal notation will be necessary, and will lead the pupils to discover recurring decimals. These provide a first encounter with infinite series which have finite sums, and are usually regarded as great fun for a time. Decimal equivalents of the simpler fractions (e.g.: $\frac{1}{2}$, $\frac{3}{4}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{5}$, $\frac{2}{5}$, $\frac{3}{5}$, $\frac{4}{5}$, $\frac{1}{3}$, $\frac{2}{3}$) should be memorised and, until decimal coinage is introduced, such simple equivalents as 1s. = £0·05, but on the whole it is advisable to use the straightforward method of conversion shown on page 18.

Multiplication. In the past, several different methods for the multiplication of decimals have been suggested and tried out. These were discussed at some length in the 1932 Report, as at that time opinion was much divided as to which was the best method. Opinion now seems to have crystallised in favour of that which treats the calculation of the figures and the establishment of their place value separately. This method is arrived at directly from the meaning of a decimal. As soon as it is realised that $2{\cdot}73=\frac{273}{100}$, and $15{\cdot}6=\frac{156}{10}$ it follows that

$$2{\cdot}73\times 15{\cdot}6=\frac{273}{100}\times\frac{156}{10}=\frac{273\times 156}{1000}=\frac{42\,588}{1000}=42{\cdot}588.$$

The numbers multiplied have 2 decimal places and 1 decimal place, the product 2 + 1 = 3 places, and so the rule is arrived at, 'Multiply the digits as if they were whole numbers and count the total number of decimal places.'

Practice with simple cases is specially important, e.g. to find $(0{\cdot}02)^3$. This has 6 decimal places of which the last is 8. The answer is 0·000 008. Similarly, $0{\cdot}04\times 0{\cdot}003=0{\cdot}000\ 12$ and $0{\cdot}037\times 0{\cdot}03=0{\cdot}001\ 11$.

Longer examples can be set down thus

$$\begin{array}{lr} & 753 \\ & 182 \\ \hline & 753 \\ 7{\cdot}53\times 18{\cdot}2=137{\cdot}046. & 6024 \\ & 1506 \\ \left\{\begin{matrix}\text{Total number of decimal places} \\ 2+1=3\end{matrix}\right\}^{*} & \\ \hline & 137046 \\ \hline \end{array}$$

The decimal point is inserted in the answer in accordance w bearing in mind that a zero which occurs at the end of the product does as one of the digits after the point even though it may subsequently be discarded, e.g.

---

* Not written down.

$$\begin{array}{ll} 2{\cdot}48\times7{\cdot}5 & \begin{array}{r} 248 \\ 75 \\ \hline \end{array} \\ =18{\cdot}600, & \begin{array}{r} 1240 \\ 1736 \\ \hline \end{array} \\ =18{\cdot}6. & \begin{array}{r} 18600 \\ \hline \end{array} \end{array}$$

Division. For division, the method of setting down the problem in 'fraction bar' form and then converting the divisor to a whole number is recommended, e.g.:

$$7{\cdot}52\div3{\cdot}7=\frac{7{\cdot}52}{3{\cdot}7}=\frac{7{\cdot}52\times10}{3{\cdot}7\ \ \times10}=\frac{75{\cdot}2}{37}$$

The second step can be left out as soon as the process is understood and the solution can be completed either by cancelling when appropriate (and a number of such examples should be taken until the idea is grasped) or by the usual methods of short or long division, e.g.:

$$\frac{3{\cdot}36}{1{\cdot}2}=\frac{3{\cdot}36\times10}{1{\cdot}2\ \ \times10}=\frac{33{\cdot}6}{12}=2{\cdot}8.$$

$$\frac{0{\cdot}0192}{0{\cdot}008}=\frac{0{\cdot}0192\times1000}{0{\cdot}008\ \ \times1000}=\frac{19{\cdot}2}{8}=2{\cdot}4$$

$$\frac{67{\cdot}2}{0{\cdot}32}=\frac{67{\cdot}2\ \ \times100}{0{\cdot}32\times100}=\frac{6720}{32}=\frac{840}{4}=210.$$

$$4{\cdot}0068\div371=0{\cdot}0108. \qquad \begin{array}{r} 0{\cdot}0108 \\ 371\,\overline{)\,4{\cdot}0068} \\ 3\ 71 \\ \hline 2968 \\ 2968 \\ \hline \end{array}$$

Find 0·007 24 ÷ 0·0892 *to three places.*

$$\frac{0{\cdot}007\ 24}{0{\cdot}0892}=\frac{72{\cdot}4}{892}\simeq0{\cdot}081 \qquad \begin{array}{r} 0{\cdot}0811 \\ 892\,\overline{)\,72{\cdot}4000} \\ 71\ 36 \\ \hline 1\ 040 \\ 892 \\ \hline 1480 \end{array}$$

Decimal places, significant figures, accuracy and approximations are considered in Chapter 13.

Standard form. A number given as $a\,.\,10^m$ is said to be in 'standard form' when a lies between 1 and 10 and m is an integer. This form is specially useful to express numbers which are either very large or very small, such as are frequently met in science; thus we may say,

'The Sun's distance from the Earth $\simeq 9{\cdot}3\times10^7$ miles,'

'The mass of an electron $\simeq 9\times10^{-27}$ grammes.'

At some stage the connection with the characteristics of logarithms should be brought out.

Notes. (1) The decimal point is written as a full stop in the U.S.A. and as a comma on the Continent; in both the multiplication dot is written as we write our decimal point. This variation in practice is illustrated below:

	British	*U.S.A.*	*Continental*
Decimal point	4·5	4.5	4,5
Multiplication dot	7 . 8	7·8	7·8

(2) Method of rough estimate:

$$23{\cdot}61 \times 42{\cdot}1 \simeq 20 \times 40 = 800.$$

$$\frac{0{\cdot}00724}{0{\cdot}0892} = \frac{0{\cdot}724}{8{\cdot}92} \simeq \frac{0{\cdot}7}{9} \simeq 0{\cdot}08.$$

When the pupil can rely on a rough approximation to fix the position of the decimal point, the point may be ignored in the working and inserted in the answer. Those who use the slide rule do this.

(3) A pupil who can recognise 0·125 as $\frac{1}{8}$ will find it easier to multiply by $7\frac{1}{8}$ rather than by 7·125.

(4) Pupils should be given a good deal of practice with multiplication and division by such numbers as 30 or 700. Simple calculations with easy numbers should be used frequently to establish fluency with basic ideas.

(5) The kind of question requiring a remainder, e.g., 'How many pieces of wire each 1·6 in. can be cut from a yard, and how much is left over?' is probably best done by changing each length into tenths of an inch; the problem then becomes 'How many pieces of wire each 16 tenths of an inch can be cut from 360 tenths, and how much is left over?' and is then a simple division.

Historical note.[*] The decimal point has an interesting history. It was not the earliest notation for representing decimals and its advantages were not immediately recognised. Indeed for a hundred years or more it was just one of a number of notations in common use. In 1543 Robert Recorde wrote the result of dividing 1253 by 10 as 125⌊3, i.e. quotient 125 and remainder 3. This was not used by Recorde as a notation for decimals but was the source of a later notation.

In 1585 the Dutchman Simon Stevin published a book to popularise decimals and he used two notations. The number 123·456 he wrote as (i) $123'4''5'''6''''$ or (ii) 123⓪4①5②6③. The decimal figures 456 were called prime, second, third, and it is interesting to compare these names with the Latin names for minutes and seconds of angle: pars minuta *prima* and pars minuta *secunda*. The modern signs for minutes and seconds may be compared with Stevin's first notation. Stevin's second notation is clearly a forerunner of index notation, and the zero indicating the unit place is particularly interesting. He is in fact writing 1, 10, 100, 1000 as 10^0, 10^1, 10^2, 10^3. Both notations are cumbersome and show no appreciation of the fact that it is only necessary to indicate the place value of one digit. Stevin may have been influenced by his desire to expound the basic principles in which place value is all important. For example in an addition he uses his symbols at the head of the columns:[†]

$$\begin{array}{cccccc} & & \textcircled{0} & \textcircled{1} & \textcircled{2} & \textcircled{3} \\ & 2 & 7 & 8 & 4 & 7 \\ & 3 & 7 & 6 & 7 & 5 \\ 8 & 7 & 5 & 7 & 8 & 2 \\ \hline 9 & 4 & 1 & 3 & 0 & 4 \end{array}$$

---

[*] Based chiefly on F. A. Yeldham, *The teaching of arithmetic through 400 years*, (Harrap).

[†] From D. E. Smith, *History of mathematics*, (Ginn & Co.), Vol. II, p. 243.

In 1631 William Oughtred went back to Recorde's symbol for division by 10 and wrote 123·456 as $123\,\underline{|\,456}$. Other writers about this time used the vertical line only, calling it the *prime line* at first and later on the *virgula* (little rod). Many writers felt the need to indicate the decimal figures and some did it by putting a dot over each one. In 1616 Kepler used 123,456 and 123(456), and Napier at about the same time used the comma and a variant of Stevin's notation, 123,4′5″6‴. In 1677 Cocker's Arithmetic mentioned that the following notations were in use and recommended the first as 'most fit for calculation': 123·456, 123 | 456, $123\,\underline{|\,456}$, $12\dot{3}456$, $12\overset{\scriptscriptstyle |}{\underset{\scriptscriptstyle |}{3}}456$, 123,456, $123,\dot{4}\ddot{5}\dddot{6}$. In the subsequent years the decimal point gradually ousted the other notations in England, but it is interesting to know that on the Continent the comma (Cocker's sixth notation) is used instead of the point (see Note 1). There is something to be said for Cocker's fourth and fifth notation. The marking of the unit digit by a dot placed after it obscures the symmetry with respect to the unit place of multiplication and division by the same power of 10, e.g.

$$2\times 10=20,\qquad 2\div 10=\cdot 2,$$
$$2\times 10^2=200,\qquad 2\div 10^2=\cdot 02.$$

With Cocker's fourth form the symmetry is obvious:

$$2\times 10=2\dot{0},\qquad 2\div 10=\dot{0}2,$$
$$2\times 10^2=20\dot{0},\qquad 2\div 10^2=\dot{0}02.$$

Many mathematicians including Augustus De Morgan (1850) have pointed out that such a notation would greatly simplify the rules for determining the characteristic of a logarithm. De Morgan advocated the rule 'The characteristic of (the logarithm of) a number is the number of places by which the first significant figure is distant from the unit's place; and is positive when that first figure falls to the left, negative when to the right'.

CHAPTER 7

PERCENTAGES

THE term *per cent* (probably derived from the Italian *per cento*) was first used in connection with interest. (Gresham in 1568 wrote of 'Th'interest of xij.per cent by the yeare'.) The modern use of percentages is much wider and the literal meaning of '12 for every hundred' has been slightly modified. The practical effect of taking 12 for every hundred is to produce an amount equal to $\frac{12}{100}$ of the original. Thus a percentage is simply a fraction with denominator 100. The advantage of expressing fractions with such a standardised denominator is that they can be easily compared one with another. The choice of 100 is of course arbitrary and for some purposes (e.g. population) 1000 is more commonly used. Decimals serve the same purpose, and percentages may be regarded as a half-way house between fractions and decimals.

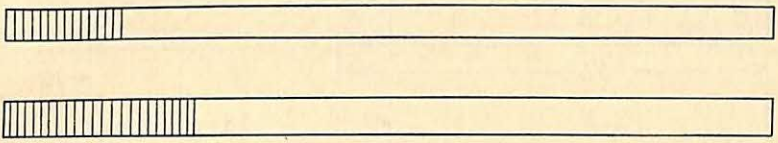


FIG. 7.1

In Fig. 7.1 the shaded parts of the two rectangles are respectively $\frac{3}{20}$ and $\frac{1}{4}$ of the areas. To compare them, it is customary to imagine the length of each rectangle to be divided into 100 equal parts and to say that the shaded portions are respectively 15 parts and 25 parts, or '15 per cent' and '25 per cent' of the whole. The decimals 0·15 and 0·25 express the same thing just as well, but the phrases '15 per cent' and '25 per cent' are commonly used and convey the meaning more clearly to many people.

At an early stage of the school work on percentages pupils should become familiar with the percentage and decimal equivalents of the simpler fractions. Diagrams may be used with advantage (Fig. 7.2).

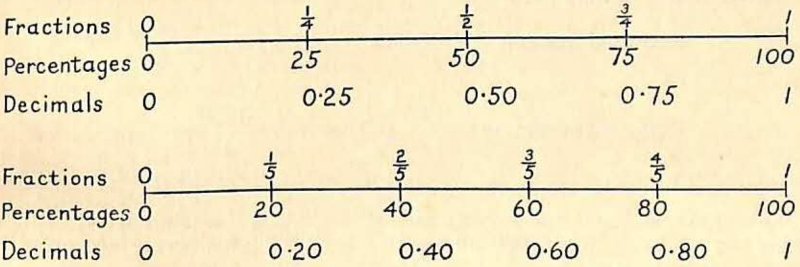


FIG. 7.2

Interchange between the fractional, decimal and percentage forms must be practised, both in simple cases taken orally and in those for which some written work is necessary. This work, with simple applications, should be allowed to become familiar before any mention is made of percentage increases and decreases.

Besides questions of the type

Find 3 *per cent of* £12

and the corresponding inverse questions, the applications should include some in which the actual amounts are not known, such as

If 63·8 *per cent of the world's oil comes from North and Central America, what percentage comes from the rest of the world?*

and some on overlapping classes, such as

If 56 *per cent of a school have had measles and* 38 *per cent have had chicken-pox and* 12 *per cent have had neither, what percentage have had both?*

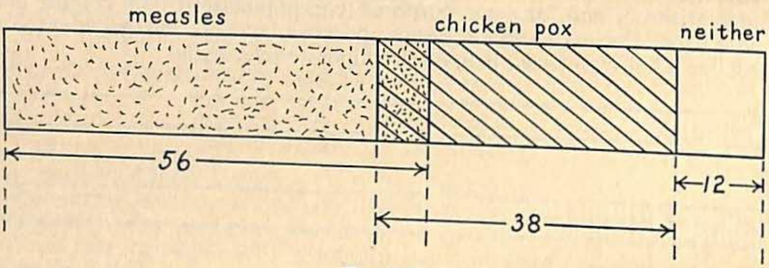


FIG. 7.3

Questions of the type

If 12 *electric lamps out of* 55 *are defective, what percentage is this?*

present some difficulty in expression. The pupil must ask himself, 'What fraction of the lamps are defective?' and should write the answer as

$\frac{12}{55}$ of the lamps are defective, i.e. 0·218 ... of the lamps.

$\therefore$ 21·8 per cent are defective.

Alternatively he may write

$$\text{Required fraction} = \frac{12}{55} = 0{\cdot}218\ \ldots = 21{\cdot}8 \text{ per cent.}$$

or

$$\text{Required percentage} = \frac{12}{55} \times 100 \text{ per cent} = \frac{1200}{55} \text{ per cent,}$$
$$= 21{\cdot}8 \text{ per cent.}$$

Questions on profit and loss should be treated on the same principle. If a man buys a house for £2000 and sells it for £2400, his percentage profit is $\frac{400}{2000} \times 100$ per cent, i.e. 20 per cent. But it should be mentioned that the shopkeeper does not reckon his percentage profit in this way. For him the retail price is commonly fixed first and is thus the 'original value'. The wholesaler allows him a discount of so much per cent on this value. The discount is his profit, and it thus becomes natural for him always to compare his profit with his selling price.

A bookseller buys a 15*s. book from the publisher at a discount of* 30 *per cent. What profit does he make on each copy?*

$$\text{Profit} = \text{discount} = \frac{30}{100} \times 15\text{s.} = 4\text{s. }6\text{d.}$$

When increases and decreases are considered (this will be at a later stage, after some knowledge of ratio has been gained), practice should first be given in the direct calculation of the increase or decrease or of the new value. The simplest cases of all, when the original value is 100 units or 1 unit, should be used to introduce the idea that, for example, an increase of 8 per cent is equivalent to multiplication by $\frac{108}{100}$ or 1·08. Results should also be expressed in the ratio form as

Original value : increase : new value
= 100 : 8 : 108,

and by means of diagrams (Fig. 7.4).

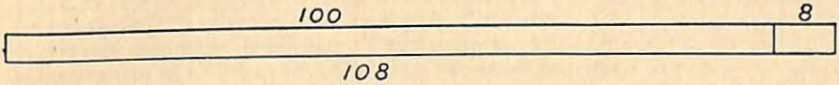


FIG. 7.4

Similarly, for a decrease of 8 per cent,

Original value : decrease : new value
= 100 : 8 : 92.

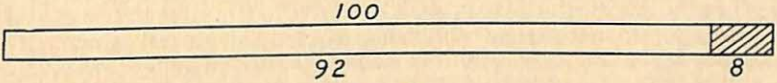


FIG. 7.5

Many oral questions should be asked of the type

Express the original value (or the increase) as a fraction of the new value.

After this, inverse questions should present less difficulty. For example

In June 1961, 230 000 *overseas visitors came to Britain, an increase of* 11 *per cent on the number for June* 1960. *How many came in June* 1960*?*

We must first ask the question, 'Per cent of what?' i.e. we must make it clear what the sentence means. The general principle is that the increase is compared with the original value, i.e. the value just before the increase; in this case, with the number for June 1960.

Original value : increase : new value
(June 1960) (June 1961)
= 100 : 11 : 111.

The number for June 1960 was therefore $\frac{100}{111} \times 230\,000$, i.e. about 207 000.

In a certain year, 5600 *copies of a book were sold, a decrease of* 20 *per cent on the previous year. How many were sold in the previous year?*

The decrease is 20 per cent of the number sold in the previous year.

$$\begin{array}{lccccc} & \text{Original value} & : & \text{decrease} & : & \text{new value} \\ & \text{(previous year)} & & & & (5600) \\ = & 100 & : & 20 & : & 80. \end{array}$$

Therefore the number sold in the previous year was $\frac{100}{80} \times 5600$, i.e. 7000.

Index numbers are an important example of percentages. For example, the index of industrial production for the United Kingdom, taking the year 1954 as 100, was 106 in 1958, 113 in 1959, and 120 in 1960. This means that in 1958 production was 6 per cent above the 1954 level, in 1959 13 per cent above, and in 1960 20 per cent above. By this means we are able to compare the production for one year with that of another more clearly than if we were given the actual values.

CHAPTER 8

RATIO AND RATE

Quantities of the same kind can be compared in two ways: by difference and by ratio. Near the end of a race a competitor is concerned with the distances that separate him from the other runners, i.e. with differences; but it would serve little purpose to say that the model of a ship was 500 feet shorter than the ship itself: it is more useful to compare them by scale, i.e. by ratio. Both these methods of comparison are in everyday use and are to that extent familiar to children. They are accustomed on the one hand to such phrases as 'two inches taller', 'sixpence more pocket money'; and on the other to 'twice as much milk', 'half as long again', 'we're half way there'.

Ratio is used whenever we answer the question, 'How many times?' To say that a football ground is $1\frac{1}{2}$ times as long as it is broad is to state a ratio; and to say that its breadth is 80 yards, i.e. 80 times the length of the standard yard, is again to state a ratio. Thus every measurement is a ratio, the given quantity being expressed as so many times a standard quantity.

A ratio compares quantities of the same kind (i.e. quantities capable of being measured in the same units), but we sometimes need to express a connection between quantities of different kinds as, for example, when we measure speed in miles per hour or state a price in shillings per hundred. These are examples of the idea of *rate*. But a rate does not necessarily connect quantities measured in different units, e.g. we speak of the standard rate of income tax being 7s. 9d. in the pound. Here the idea is that, for every pound of income in excess of a certain amount, 7s. 9d. tax is to be paid. Rates generally include the phrase 'in the', the word 'per' (from mediaeval Latin, meaning 'for every' or 'for each') or, more briefly, 'a', implying that the magnitude of one quantity will be determined from the other, using the given rate. Thus the price of eggs 3s. 9d. a dozen implies that 2 dozen will cost 7s. 6d., $\frac{1}{2}$ dozen 1s. $10\frac{1}{2}$d., and so on. In short, a ratio is used to connect two quantities, but a rate implies a functional relationship and is used to connect corresponding values of two variables.

Consider for example the following sets of numbers:

Distance in yd.	50	5050	7050	10 050
Clock time (a.m.)	11.00	11.05	11.07	11.10

No comparison between these two sets has any meaning until we know of some connection between them which enables us to pair them off in a (1, 1) correspondence. For example, if the upper set gives the distance of a vehicle from a fixed point at the times given by the lower set, then we can match the sets as shown in the following diagram:

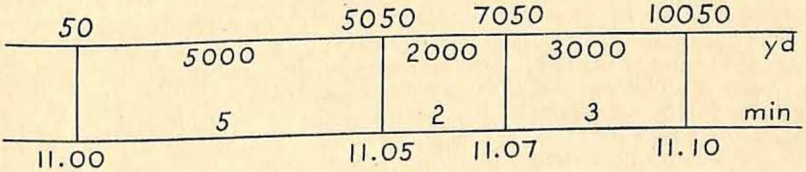


Fig. 8.1

Considering any two intervals in one line, and the corresponding intervals in the other line, we discover an equality of ratios:

$$\frac{5000\text{ yd}}{2000\text{ yd}}=\frac{5\text{ min}}{2\text{ min}};\quad \frac{2000\text{ yd}}{3000\text{ yd}}=\frac{2\text{ min}}{3\text{ min}};\quad \frac{3000\text{ yd}}{5000\text{ yd}}=\frac{3\text{ min}}{5\text{ min}}$$

But we can also match corresponding intervals across the lines as follows:

$$\frac{5000}{5}=\frac{2000}{2}=\frac{3000}{3}=1000$$

We say that this indicates that distance is covered at the constant rate of 1000 yd per min.

An equality of ratios is described as a *proportion*. In the table

Income	£500	£600	£700
Tax	£100	£120	£140

we note that $\frac{£600}{£500}=\frac{£120}{£100}$ and that $\frac{£700}{£600}=\frac{£140}{£120}$. We describe this by saying that the tax is proportional to the income. It is also seen that $\frac{£100}{£500}=\frac{£120}{£600}=\frac{£140}{£700}$, i.e. the rate of tax is constant. Thus when two variable quantities are in proportion, there is a constant rate as well as equality of ratios.

In everyday language the word proportion is used with different meanings in such phrases as 'a proportion of the cost' and 'the proportions of the human body'. Time is well spent in examining such usages and contrasting the mathematical and more general uses of the word.

Examples of the type *Find the cost of* 3 *yards at* 1*s*. 9*d*. *a yard* are examples of proportion. Such questions, and those of the inverse type *If tennis balls cost* 32*s*. *per dozen, find the cost of each* can be treated as mere multiplication and division. Thus ratio, rate and proportion are used informally at a very early stage of the arithmetic course. The compiling of a ready reckoner, showing perhaps the cost of 1, 2, 3, 6, 12, 24, 36 tennis balls, implants the idea of the relationship between two variables. The ready reckoner also provides an informal introduction to the idea of ratio, for in the course of its construction it will be at once evident that the easiest way to find the cost of 24 balls is to double the cost of 12; and, when the table has been completed, it can be pointed out that 36 is 3 times 12, or 1½ times 24, and that the table can be checked at any point by comparing one reading with another in this way.

In both the examples of the last paragraph the rate is assumed to be uniform and a straight line graph appears if the readings in the table are plotted on squared paper; but it is undesirable that all the graphs in the early stages should be linear. An example of a non-linear graph, for which the data can be collected by the pupils, is the flow of traffic in vehicles per hour. Graphs of this type help to bring out the important fact that rates need not be uniform. In contrast the straight line graph comes to be recognised as the pattern of a constant rate of growth or change, and its slope is seen to give a measure of the rate. The graph showing the number of apples given to 1, 2, 3, 4, . . . children at the rate of 3 apples per child is similar to that showing the number of gallons of water which have flowed from a tap in 1, 2, 3, 4, . . . minutes at the rate of 3 gallons per minute; that the latter may be used to give intermediate values, while the former cannot, is a simple illustration of the important ideas of continuity and discontinuity.

Questions of the type

If 6 *eggs cost* 1*s*. 9*d*. *find the cost of a dozen.*

If 100 *identical castings weigh a ton, what would* 25 *weigh?*

may be taken at this stage, as well as some in which it is natural and proper to deal with one unit as a first step, e.g.

If a motor coach is hired for 36 *people at a cost of* £9, *find the cost per head and the charges for groups of* 5, 6 *and* 8.

If an aircraft covers 200 *miles in* 25 *minutes, find its speed in miles a minute, the distance it would travel in* 5 *seconds, and the time it would take for* 5 *miles.*

It will be noticed however that, in the simple examples so far mentioned, there is no difference in method, or in degree of difficulty, between those in which the unit step is used and those which are done directly. The process is simply a multiplication or a division.

After this, perhaps at a distinctly later stage, a more formal treatment of ratio and proportion will be undertaken and the appropriate technical terms introduced. The teacher's first task will now be to make the idea of ratio explicit: for this, visual aids may well be used. Two lines on the blackboard can be compared roughly by ratio without their actual lengths being known,

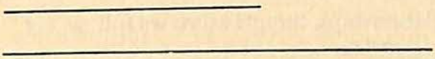


FIG. 8.2

Heights, preferably distant heights, can be compared. The proportions of the human body can be studied, (in Fig. 8.3, the man's height is h units), and it is worth while seeking out examples of ratio in which the actual measurements

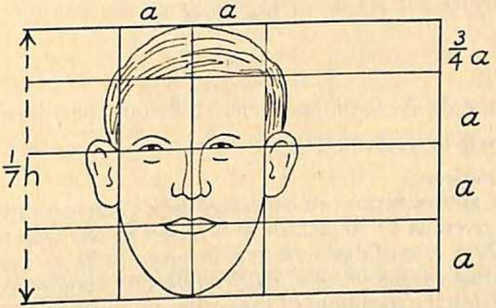


FIG. 8.3

are not known, e.g. the apparent distances between stars, the length and breadth of Australia, the length and height of a locomotive seen in a picture. These rough comparisons will be followed by more accurate ones in which the actual measurements necessarily enter, and the ratio notation will now be introduced. The statement '$A : B = 3 : 4$' means that if A is divided into 3 equal parts then B can be divided into 4 parts of the same size.

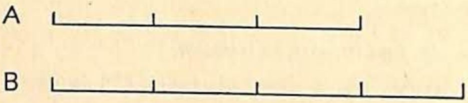


FIG. 8.4

Hence it implies a two-way relationship: A is $\frac{3}{4}$ of B, and B is $\frac{4}{3}$ of A. A similar statement for a three-term ratio, such as '$a : b : c = 3 : 4 : 5$' (for the sides of a triangle, for example)

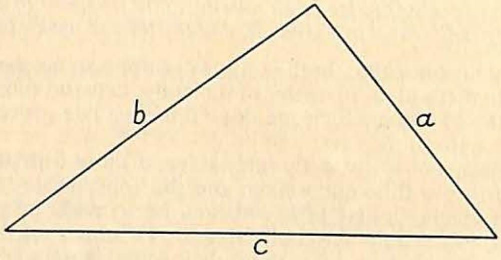


FIG. 8.5

implies six such relationships. Simple exercises follow, e.g.

If $b = 20$ *cm, find a and c.*

These may be done informally, with the aid of sketch diagrams. Such a question as

Divide 24 *cm in the ratio* 3 : 4 : 5.

may be taken at this stage, a diagram being used to show

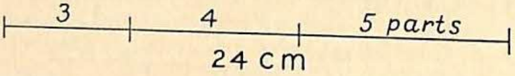


FIG. 8.6

that the parts are $\frac{3}{12}$, $\frac{4}{12}$, $\frac{5}{12}$ of the whole. This would be followed by

Divide 20 *cm in the ratio* 3 : 4 : 5.

and similar questions.

In practical affairs ratios are expressed in a great variety of ways. A map scale may be given as 1 : 10 560, or as 6 *inches to one statute mile*, or as 880 *feet to one inch*. A rate of discount may be given as 7½ *per cent* or as 1*s*. 6*d*. *in the £*. Such phrases as *one-third as much again* are in common use. Time is well spent on clarifying the meanings of these alternative forms.

Reference should now again be made to the ready reckoner, and examples in proportion should be given. These should begin with whole-number multipliers and proceed to easy fractions as, for example, in

If it takes 40 *minutes to fold* 1000 *pamphlets, how long would it take to fold* 2000*? How long to fold* 1500*?*

If 150 *hymn-books cost* £9 7*s*. 6*d*. *find the cost of* 100.

If a man walks 4 *miles in an hour, how far will he walk in* 40 *minutes?*

It is unwise to impose the fractional method on pupils who cannot use it easily. It is better to let them pass through the intermediate stage of finding the cost of 50 hymn-books or even of one. Later they may invent the fractional method themselves as a short cut. There are of course many examples in which it is of interest to know the cost of one article, and for these the unitary method is appropriate. It should not be developed into a routine. The compiling of a ready reckoner, in which the cost of one article will naturally be shown, brings out also the fact that it is easy to pass from one reading to another without going through the 'unit step' each time. The writing out of questions in proportion is a matter in which pupils need varying degrees of help. First they must be induced to write a sentence stating the fact given; then a similar sentence, with a blank in it, for the fact to be found. This form, because it sets out the comparison between the two cases, is instructive and helpful. It appears sooner or later that it is advisable to look ahead when composing the first of these sentences and to arrange it so that the unknown quantity will be at the end of the second sentence.

The idea of matching two sets of numbers represented by points on two straight lines can be useful with pupils who have difficulty with the English.

If 20 *oranges cost* 4 *shillings, find the cost of* 15 *oranges.*

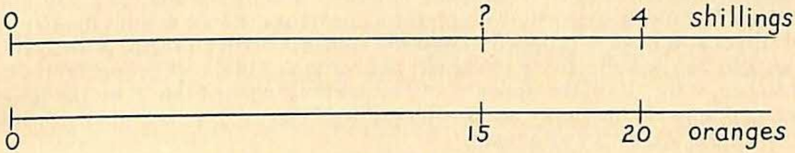


FIG. 8.7

We have a choice of two methods corresponding to the two methods of comparing the four intervals:

(i) The number of oranges changes in the ratio $\frac{15}{20}$. Therefore the required price is

$$4 \times \frac{15}{20} \text{ shillings} = \dots .$$

(ii) We pay at the rate of $\frac{4}{20}$ shillings per orange. Therefore the cost of 15 oranges is

$$15 \times \frac{4}{20} \text{ shillings} = \dots .$$

The first method uses ratio, and the second rate. In the Unitary Method, we say

$$\begin{aligned} &\text{20 oranges cost 4s.,} \\ \therefore\ &\text{1 costs } \tfrac{4}{20}\text{s.,} \\ \therefore\ &\text{15 cost } 15 \times \tfrac{4}{20}\text{s.} \end{aligned}$$

This uses ratio, because we are comparing the number of oranges, but the rate appears too, in the intermediate step.

In general the more complicated examples should be avoided. Questions in which fractional quantities have to be compared, e.g.

If $1\frac{1}{2}$ *yards of material cost* 23s. 9d. *what would* $2\frac{1}{3}$ *yards cost?*

may be treated by reducing to a smaller unit (e.g. 6 inches).

Some attention should be given to problems in which the quantities are not in direct proportion, e.g.

If it takes 4 *minutes to boil an egg, how long does it take to boil two eggs?*

If a pound of butter lasts 3 *people for a week, how long will it last* 6 *people?*

If a man has £200 *when he is* 20, *how much will he have when he is* 60?

If a halfpenny weighs $\frac{1}{5}$ *oz. how much does a penny weigh?* ($\frac{1}{3}$ *oz.*)

If a boy walks to school in half an hour and can cycle three times as fast, how long does he take to cycle?

If a metal plate 3 *inches square weighs* 6 *oz. what would be the weight of one of the same thickness* 6 *inches square?*

It is at this stage that the terms *proportion, direct proportion* and *inverse proportion* are needed. Further tables should be compiled (on the ready reckoner basis) and graphs drawn from them. A money-quantity graph for a constant buying rate and a speed-time graph for a constant distance exhibit the nature of direct and inverse proportion and the contrast between them. A slide rule can also be used effectively to exhibit two variables in direct proportion. For instance, if the '1' on the upper of two scales is set against the '2' on the lower scale, points on the upper scale will read half the value of the points on the lower scale in contact with them.

Some correlation should be established with scale drawing and work on similar triangles. The introduction of trigonometry is thus assisted and trigonometry can be used in turn to illustrate and strengthen the knowledge of ratio already acquired. Some classes may be able at this stage to tackle the law of squares for areas of similar figures, followed by that of cubes for volumes.

A ready reckoner or a table for a direct proportion graph may be extended in either of two ways; we can proceed from one reading to another by ratio (multiplying the number of articles and the cost by the same number); or we can find the new reading by knowing the rate (multiplying the cost per article by the number of articles). For inverse proportion (e.g. the times taken for a journey at different speeds) there is a similar choice: we can alter the two quantities in inverse ratio (multiplying one and dividing the other by the same number); or we can use the fact that the product of the two quantities is constant. The interpretation of this product in particular examples is of some interest: it may in one case be the length of the journey, in another the measure of a piece of work in man-hours. This may lead up to the laws $y=kx$, and $xy=K$ or $y=\frac{K}{x}$, and these should be introduced or illustrated by graphs.

Such compound units as *man-hours* are useful for questions of the type

If a job can be done by 12 *men in* 7 *working days, how much longer will it take if* 5 *of them stop work after the first* 4 *days?*

Other good examples of these product-units are the 'ration' (e.g. one man's food for a week) and the 'unit' of electricity (defined as a kilowatt-hour). Questions in compound proportion are apt to be artificial and should not be made too elaborate. In commercial practice such computations are often avoided or simplified by the use of tables; the construction and use of such a table can sometimes be more illuminating and interesting than working a series of miscellaneous examples.

The idea of increasing or decreasing in a given ratio may well be introduced in connection with ready reckoner tables or with graphs, and should be thought of at first simply as multiplying by a fraction greater or less than 1. The phrase *increase in the ratio* 5 *to* 4 may be confusing to beginners and can be replaced at first by *increase in the* ratio $\frac{5}{4}$. The use of ratios and multipliers is more than a piece of arithmetical technique. The habit of thinking in such terms enlarges the mathematical outlook, carrying the pupil forward to trigonometry, to variation and to the idea of functionality. The teaching of ratio in arithmetic should be directed to this end.

Once pupils have acquired an understanding of and a facility in the use of ratio and rate, the contexts in which they are required to apply the ideas can be enlarged. These contexts will depend on the pupils' vocabulary and experience; often their success will depend as much on their full understanding of the various terms used in a particular sphere or topic,—e.g., the rateable value of a property or the percentage yield on an investment—as on their ability to carry out the calculations involved. The teacher's role is to encourage this full understanding by providing numerous questions involving the terms and requiring only simple calculations; a few questions taken from the adult business world are likely to stimulate interest, but too many may prove time-wasting.

Moving from one topic to another enlarges the vocabulary of the pupils, and increases the range of problems they can tackle without necessarily demanding a different arithmetical process. Generally the principle of direct proportion is used.

Among topics introduced may be some of the following; the order in which they are placed is not necessarily the order in which they should be taken.

Discount.
Simple interest.
Rate of profit or loss.
Depreciation in value.
Compound interest.
Income tax.
Hire purchase.
Local Authority finance.
Insurance.
Investments (shares and stocks).
Pressure.
Speed.
Density.
Household arithmetic.
Monetary exchange.

The problems which arise may be 'direct' or 'inverse'; the latter are the more difficult, and some guidance in the method of attack is called for. As stated earlier in this chapter, the writing of a statement in a particular way is often the key; the following outlines of worked examples serve as illustrations. They should *not* be taught to pupils as methods which they must use; but rather, each pupil should be encouraged to develop his own style subject to the requirements of orderly presentation and accurate calculation.

Simple interest

Direct: Find the interest on £350 *for* 7 *months at* 4% *p.a.*

Two possible approaches to the solution are:

(i) Interest on £100 for 1 year $= £4$.

$\therefore$ interest on £350 for 7 months $= £4 \times \dfrac{350}{100} \times \dfrac{7}{12}$.

(ii) Interest on £350 for 1 year $= \dfrac{4}{100} \times £350$, etc.

Inverse: What sum of money invested at 4% *p.a. amounts to* £358 3*s*. 4*d*. *in* 7 *months?*

The statement needs to end with the sum invested so it will help to know the result of investing £100.

$$\text{Interest on £100 for 7 months} = £4 \times \frac{7}{12},$$
$$= £2\tfrac{1}{3}.$$

$\therefore$ £$102\frac{1}{3}$ is the amount when the principal is £100.

$\therefore$ £$358\frac{1}{6}$ is the amount when the principal is £$100 \times \dfrac{358\frac{1}{6}}{102\frac{1}{3}}$, etc.

Some teachers may prefer to use algebra in which case the question may be answered as follows:

Let the sum invested be £x.

$$\text{Interest on £}x\text{ at 4\% for 7 months} = £x \times \frac{4}{100} \times \frac{7}{12},$$
$$= £\frac{7x}{300}.$$
$$\therefore\ x + \frac{7x}{300} = 358\tfrac{1}{6}.$$
$$\therefore\ \frac{307}{300}x = \frac{2149}{6}.$$
$$\therefore\quad x = \frac{7 \times 300}{6},$$
$$= 350.$$

$\therefore$ the sum to be invested $= £350$.

Profit and loss

In the direct case it is often better to calculate the gain (or loss) and then to add (or subtract) than to use the factor $\left(1 + \dfrac{r}{100}\right)$. E.g. if the question were

An article bought for £7 18*s*. 4*d*. *was sold at a loss of* 3%; *what was it sold for?*

it would be better to calculate 3% of £7 18*s*. 4*d*. and to subtract than to find 97% of £7 18*s*. 4d.

Inverse: By selling sugar at 9*d*. *a lb a grocer gained* 26% *of the purchase price. What did* 1 *cwt of sugar cost him?*

He sold 1 cwt of sugar for $\frac{3}{4} \times 112$s. i.e. for 84s.
He gained 26% of cost price.

$$\therefore \text{ selling price } = \frac{126}{100} \text{ of cost price.}$$

$$\therefore \text{ cost price } = \frac{100}{126} \text{ of selling price.}$$

$$\therefore \text{ cost price of 1 cwt } = \frac{100}{126} \text{ of 84s.} = \text{66s. 8d.}$$

Depreciation

Suppose that the value of a new car depreciates by one-sixth of its value during its first year, and in subsequent years by one-eighth of the value at the beginning of the year.

Direct: A new car cost £1200; *find its value after* 2 *years.*

$$\text{Value after 1 year } = £1200 \times \frac{5}{6};$$

$$\therefore \text{ value after 2 years} = £1200 \times \frac{5}{6} \times \frac{7}{8} = £875.$$

Inverse: The value of a car after 2 *years is* £875; *find its original value.*

$$\text{Value after 1 year } = £875 \times \frac{8}{7}.$$

$$\therefore \text{ original value } = £875 \times \frac{8}{7} \times \frac{6}{5} = £1200.$$

Familiarity in forming the necessary mutlipliers, particularly in the inverse cases, is important.

Compound interest

Compound interest operates when the account is made up at regular intervals in such a way that the interest as it becomes due is added to the principal. This is the case in the calculation of mortgages and savings accounts. In actuarial theory, as applied to annuities and life assurances, the money invested is regarded as being increased continuously in such a way as to result in a certain annual percentage increase.

Such questions as *Find the compound interest on* £395 *for* 3½ *years at* 4 *per cent* cannot be answered without instructions as to how the account is made up. In practice, if the account is made up annually, interest might not be earned for the last half-year. If the account is made up half-yearly by the addition of 2 per cent, the question should be framed accordingly. (The effective annual rate is then 4·04 per cent.) The theory of continuous growth gives the amount as $£395 \times 1{\cdot}04^{3\cdot 5}$.

Questions in compound interest provide exercise in:

1. straightforward computation, using a compact arrangement of working;
2. logarithms, using the formula $A = PR^t$, where $R = 1 + \frac{r}{100}$;
3. making and using tables, including those used for annuities, etc.

Local Authority finance: Rates

The pupil should not only learn that local 'rates' are paid at so much in the pound on the 'rateable value' of each house, but should also look at this payment from the receiving end, that of the local authority. Here are two examples which illustrate the type of question which may be given to the pupils.

Example 1. *The rateable value of a house is* £56; *how much must the householder pay each half year, if the rate is fixed at* 17*s.* 2*d. in the* £ *per annum?*

Example 2. *A new village hall estimated to cost* £8700 *is to be built. Compensation and a ministry grant will provide* £6200 *of the money needed and the rest is to be paid for by an extra rate of* 2*d. in the* £1. *What is the total rateable value of the village?*

If the actual cost of the hall increases to £9950 *and the extra cost is borne entirely by the rates, how much extra will a certain villager pay if the rateable value of his property is* £60*?*

Speed

Example 1. *A cyclist takes* 15 *minutes to get to school when riding at an average speed of* 10 *m.p.h. On the return journey by the same route the average speed is reduced to* 6 *m.p.h. What is the average speed for the double journey?*

Average speed $= \dfrac{\text{total distance}}{\text{total time}}$. (Units: miles, hours)

Distance from school $= 10 \times \frac{1}{4}$ miles $= \frac{5}{2}$ miles.

Time for outward journey at 10 m.p.h. $= \frac{1}{4}$ h.

$\therefore$ time for return journey at 6 m.p.h. $= \frac{10}{6} \times \frac{1}{4}$ h $= \frac{5}{12}$ h.

$\therefore$ average speed

$$= \frac{5}{\frac{1}{4} + \frac{5}{12}} \text{ m.p.h.,}$$

$$= \frac{5 \times 12}{3 + 5} \text{ m.p.h.,}$$

$$= 7\tfrac{1}{2} \text{ m.p.h.}$$

Example 2. *If the maximum speed of a car* 16 *feet long is* 70 *m.p.h., what is the least time it will take to pass a transport lorry* 30 *feet long moving in a parallel traffic lane at* 45 *m.p.h. and how far will it travel while doing so?*

Rate of overtaking $= (70 - 45)$ m.p.h. $= 25$ m.p.h.

$$= \frac{25 \times 88}{60} \text{ ft per sec,}$$

$$= \frac{5 \times 22}{3} \text{ ft per sec.}$$

Distance to be gained $= (16 + 30)$ ft $= 46$ ft.

Time taken to pass $= \dfrac{\text{distance}}{\text{speed}}$,

$$= \frac{46 \times 3}{5 \times 22} \text{ sec,}$$

$$=\frac{69}{55}\text{ sec,}$$

$$=1{\cdot}3\text{ sec,}$$

correct to nearest $\frac{1}{10}$ sec.

$$\begin{array}{r} 1{\cdot}3 \\ 55\,\overline{)\,69} \\ 55 \\ \hline 140 \\ 165 \\ \hline \end{array}$$

Distance moved in this time = speed × time,

$$=70\times\frac{88}{60}\times\frac{69}{55}\text{ ft,}$$

$$=129\text{ ft, correct to nearest ft.}$$

Variable rates

Speeds are, in practice, seldom uniform. We therefore are concerned in elementary stages with *average* speeds. A further important development is the consideration of *variable* speeds and, in general, variable rates. The pupils should realise that a steeper slope on a distance-time graph implies a greater speed and, in the later stages of the arithmetic course, the estimation of the speed at an instant by drawing a tangent helps to prepare the way for the calculus. This is treated more fully in the Association's Report *The teaching of calculus in schools*.

A graph of the Amount at compound interest at, say, 10% is also instructive: while the rate of interest is constant, and there is a constant growth factor of 1·1, the rate of growth of the Amount increases with time. This graph reappears in an algebraic setting at a later date.

CHAPTER 9

MENSURATION

MENSURATION has been described as one half of arithmetic. It is the application of arithmetic to geometrical aspects of concrete things which are easily open to observation. Its successful pursuit involves geometrical perception developed by such experiences as drawing, measuring and modelling, and by observation. The extent of actual practical exercises at the secondary stage depends on the experience of pupils in the primary years, but they can seldom be omitted altogether.

It is essential that clear conceptions of units of measurement should be in the pupil's mind and that the units stressed are those in common use. Some account of the history of the commoner English measures, e.g. the furlong, the acre, is usually of interest. Metric as well as English measures should be included, also some metric equivalents of English measures, for instance, 1 inch $=2{\cdot}54$ cm, 1 metre $\simeq 39{\cdot}37$ in., 1 km $\simeq \frac{5}{8}$mile.

Many of the geometrical concepts relating to areas and volumes are likely to have been accepted from experience before the secondary stage. But difficulties in applying arithmetic to measured lengths and shapes will arise, as, for example, in considering the area of curved surfaces, the relations between volume and capacity, the possible equality of surfaces or volumes of differing shapes. These all require geometrical as well as numerical treatment.

Length. Pupils at the secondary stage have probably had experience of measuring lengths and of the commoner units employed for this purpose, but the range of this experience will vary greatly. Many of the suggestions contained in the following notes may have received some preliminary treatment, but adequate appreciation of the considerations raised cannot be assumed.

(i) Estimation of a length may be treated as a first crude measure of it. The individual should carry in his mind useful standards for comparison in estimating heights and distances. Thus a height may be estimated as 'About 3 times the height of the door, i.e. about 20 ft', or the distance of an object may be estimated as 'Twice the length of a cricket pitch, or between 40 and 50 yd'. For the shorter standards, multiples up to perhaps six times may be used, but if the distance is a long one, a near-equivalent or a longer standard may be used, for example 'A few yards further than the distance from the school door to the school gate', or 'Rather more than half the length of a netball court'. Thus the use of estimation brings out clearly the notion of ratio implicit in all measurement.

(ii) The application of individual measures such as the pace, the foot-length, the span and the finger-joint may serve as a second approximation. Interest may be aroused in bodily proportions, for instance in the usual equality of height and arm span. The finding by an individual of his average pace is valuable both as a practical exercise and as an established result for future use.

(iii) Longer distances are more difficult to estimate and to measure than short ones. Numerical relationships such as 1760 yards to the mile may have little meaning in terms of reality even when related to some known measured mile. The techniques of measuring distances beyond the limits of the school buildings are interesting and mathematically informative.

(iv) A curved length is more difficult to measure than a straight segment of a line. The use of a variety of measuring instruments for finding a curved length, e.g. a flexible string or tape-measure and a rolling wheel, provides useful data for discussing the approximate nature of measuring.

(v) All measuring is approximate. The round numbers given in text-book examples mask this, for they are idealised or hypothetical measures, comparable with the suppositions made in geometry, such as '*AB* is a straight line', or with the simplifications used in all branches of science, e.g. 'uniform density', 'rigid body', 'pure oxygen'. But round numbers may also be approximations and then it becomes important to know whether '13 ft' means '13 ft to the nearest foot' or '13·0 ft' or '13·00 ft.' This is not easy for a pupil to appreciate unless it is carefully related to his experience of measurement. If measuring is confined to man-made objects, such as tables and rooms, numbers which appear to be round numbers are likely to occur. These reflect the maker's skill in working to a specification and they provide an opportunity for such questions as 'How close is it?' 'Do you mean "18·0 inches" or "18·00 inches"?' On the other hand the collection of natural statistics such as pupils' heights, the heights of plants, or the measuring of outdoor distances, will more obviously demand that pupils should work to some chosen degree of accuracy.

In the collection of statistics results may have to be given in categories, for example heights of 5 ft and less than 5 ft 1 in., 5 ft 1 in. and less than 5 ft 2 in. etc. Individual results should always be stated in some form related to the degree of accuracy required, such as 'correct to the nearest inch' or 'within $\frac{1}{16}$ inch'.

(vi) In general 1 or 2 per cent accuracy is a reasonable standard for school measuring. This standard is given by the ratio of allowable error to the measure itself. Correct 'significant figures' are therefore important and correct decimal places are not necessarily a guide to what is relevant or possible.

(vii) The degree of accuracy of a measure depends both on the instrument used and on the user. Parts of a unit expressed in decimal fractions conveniently exhibit degree of accuracy. Thus measuring in inches and tenths of an inch, and estimating tenths of a tenth, is an activity appropriate to the learning of decimal fractions.

(viii) Consistency is not to be expected in the results of a number of independent trial measurements of a length. What is important is that inconsistencies should not exceed reasonable tolerances.

(ix) The value of improving standards of accuracy by refinements in the instruments used may at later stages be discussed in relation to the purpose of the work. This kind of consideration may arise in the science room, where calculations may include measured results in other fields in which equal standards of accuracy are not possible.

(x) To know the standards of accuracy used in measuring lengths is important when dealing with the mensuration of areas and volumes.

Area. Discussion of the concept of area and its measurement has probably been begun in the primary school, but revision of first principles is recommended. It is a novel idea to have to choose both the size and the shape of the unit of measurement. The convenience of the square as an intermediary in the comparison of two areas of different shapes is exhibited, for instance, in the square grid on a large-scale map (though this is mainly used for fixing position). The fact that a square inch of area is not necessarily of square shape, i.e. an inch square, has to be noted.

The determination in square units of any enclosed area, e.g. the surface of a leaf, by counting squares and part squares on a superimposed grid is an interesting exercise in itself and a useful approach to revision of any rules already learned for finding the areas of particular shapes, including the rectangle.

Transparent squared paper, obtainable in various rulings, may be used as a grid. Shapes can be duplicated in various ways, or found in atlases and other books so that several pupils may make and compare estimates of the same area. Shifting the grid to a new position on the shape to be measured gives a new count of squares and part squares. (An alternative is to use squared paper and to duplicate shapes on to it. This can be done with a spirit duplicator or with a map roller which can often be borrowed from the geography department.) Estimates of part squares can be made in various ways, perhaps beginning with counts of parts more and less than a half.

The fundamental process of square counting, even when applied to the simple case of the rectangle, is worth establishing as a mental picture available in all cases of difficulty. Moreover, the area of a rectangle provides a diagrammatic representation of multiplication. A series of rectangles, drawn to scale if necessary, such as:

22 ft by 25 ft
3 in. by $2\frac{1}{2}$ in.
$3\frac{1}{4}$ in. by $2\frac{1}{2}$ in. (Fig. 9.1)
2·3 in. by 0·3 in. (Fig. 9.2)

can be used to illustrate the rules appropriate for multiplication of integers and of vulgar and decimal fractions.*

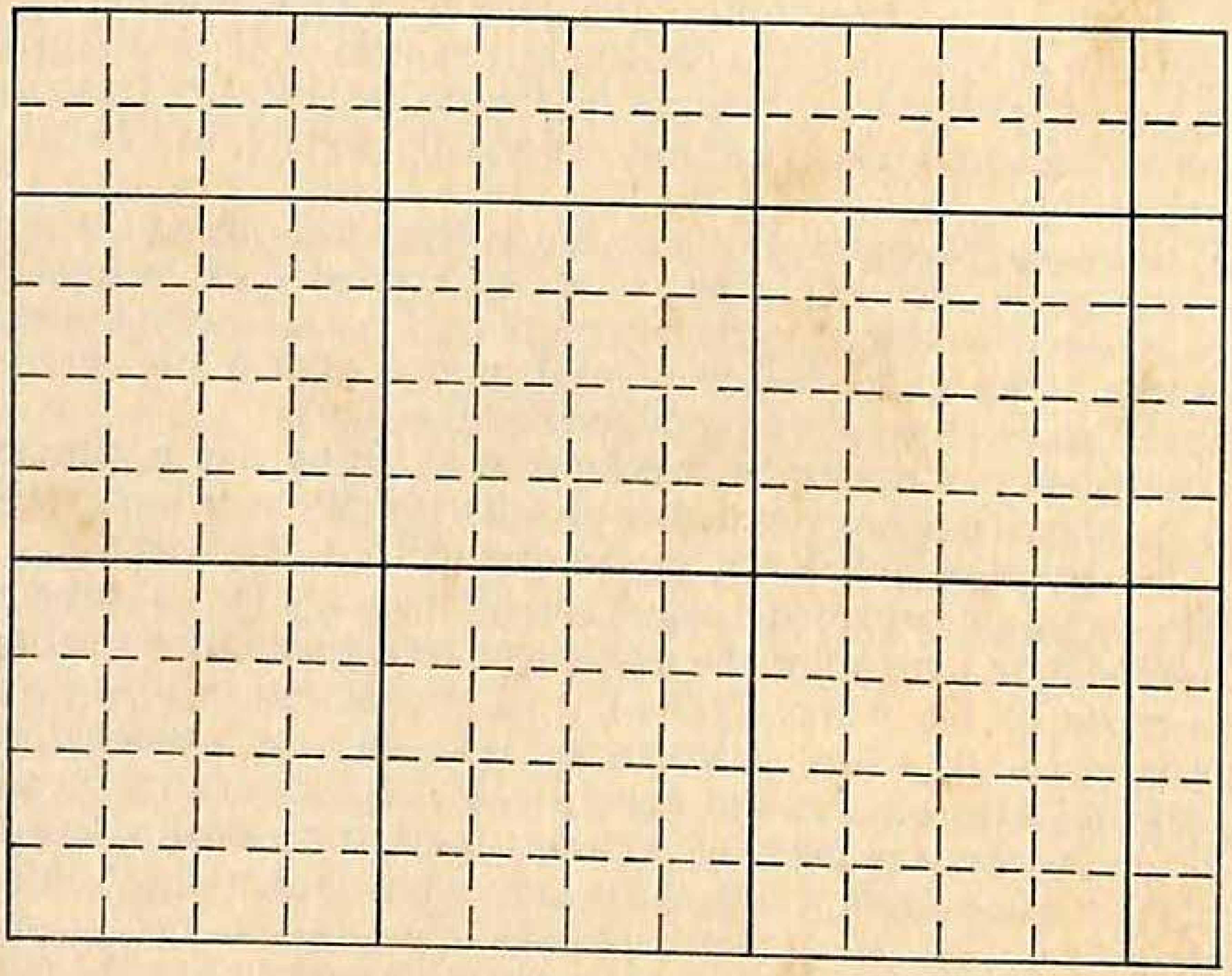


FIG. 9.1

---

* See *Mathematics in secondary modern schools*, 4.812, 6.246, 6.248, also Chapter 6 of this Report.

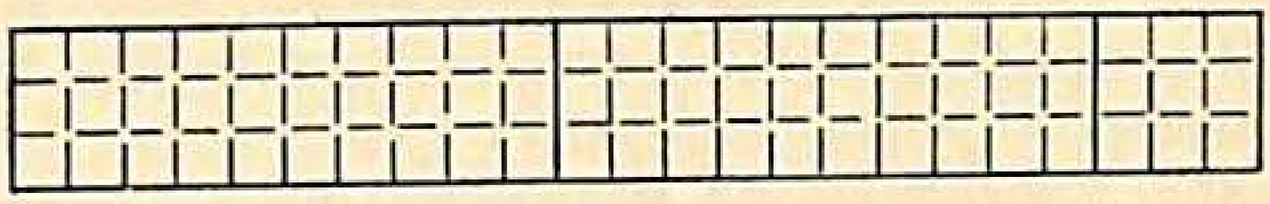


FIG. 9.2

Difficulties with units which often occur when area is treated purely arithmetically are resolved by an appeal to a squared diagram. For instance, an area which is 2 feet by 5 inches must be treated either as

24 inches by 5 inches with an answer in square inches
or 2 feet by $\frac{5}{12}$ feet with an answer in square feet.

If the mistake is made of saying 2×5 units, the units are not squares but rectangles 1 foot by 1 inch.

Further points to be noted are:

(i) Numerical and graphical work on the areas of squares with their direct bearing on tables of square measure should be included.

Thus a square of side 3 feet contains 3×3 square feet;
a square of side 1 metre contains 100×100 square centimetres;
a square of side 1 inch contains 4×4 squares of side $\frac{1}{4}$ inch,
or a square of side $\frac{1}{4}$ inch is $\frac{1}{16}$ square inch.

(ii) The expression ‘120 yards by 80 yards’ is commonly written ‘120 yards × 80 yards’; also, for instance, a carpet may be marked ‘8 ft × 12 ft’. In this context the symbol ‘ × ’ stands for the word‘ by’ and not for multiplication, and it is better not to use ‘129 yards × 80 yards’ in statements of steps in calculation. Confusion may be avoided by making a practice of stating units used before beginning calculations. For instance, the answer to the question, ‘What sized squares shall we use?’ might give rise to the following form of statement:

Find the area of a piece of wood 3 *feet by* 5 *inches.*

Length $= 3$ ft. (Working in feet and square feet.)

Breadth $= \frac{5}{12}$ ft.

Area $= 3 \times \frac{5}{12}$ sq. ft $= 1\frac{1}{4}$ sq. ft.

(iii) From the use of a squared grid for finding the area of any shape the idea readily emerges that a similar shape with double the linear dimensions has four times the area, and in general that areas of similar shapes ‘grow’ by squaring the ratio of their sides.

(iv) There is a well-known class of problems about the areas of paths bordering lawns, the volume of wood required to make boxes, the weight of metal in pipes, etc. There is no particular difficulty in the method based on subtracting from the whole the area, or areas, not required, but considerable resistance to it is encountered in the early stages of the secondary school from pupils who have been accustomed to dividing the border of a rectangle into a number of rectangles. Diagrams may not give enough help to pupils who are still having difficulty with the idea of subtraction of areas. Pieces of card which can be moved about, superimposed on one another in different positions and, if necessary, cut up may help to make things clear. The result of cutting out a corner piece from a rectangle can be compared with the result of cutting out an equivalent piece from inside.

(v) From the area of a rectangle are derived successively rules for the area of the parallelogram, triangle and trapezium. It should be emphasised that any side of a parallelogram or triangle may be taken as 'base'. Particular attention should be paid to those figures in which the altitude falls outside the base. It is a good exercise to calculate the altitude to the hypotenuse of a right-angled triangle by equating the area of the triangle found by using the two sides containing the right angle as base and height to that of the triangle taking the hypotenuse as base. For the area of a trapezium pupils should eventually derive the formula $\frac{h(a+b)}{2}$ and appreciate that its use shortens computation.

(vi) The mensuration of the circle is considered later.

Volume. The fundamental difficulties in dealing with measurement of volume are geometrical rather than arithmetical. It is essential that each individual should hold clearly a few basic ideas.

(i) The unit of volume is a cube with side of unit length. Cubic units may be built up into rectangular solids and the number of cubes in the solid can be considered as the number of cubes standing on one face multiplied by the number of such layers.

This may be reduced to the continued product of the numbers of cubes along a length, a breadth and a height.

(ii) If the numerical difficulties in dealing with areas, e.g. in areas involving fractions of a unit, have been dealt with thoroughly, a very simple treatment of analogous difficulties in volume finding should suffice.

(iii) A cubic yard contains 3 × 3 × 3 cubic feet. A box of this size and shape holds a surprising amount: over 168 gallons or about a ton of gravel.

(iv) Units of volume may be related to units of capacity found by filling one vessel with the contents of another. Thus 1 litre = 1000 cubic centimetres. A cubic foot holds about 6¼ gallons, etc.

(v) Capacities are difficult to estimate and to measure directly. One gallon is legally defined as the volume of 10 lb of water. (See Appendix.) Another useful equivalent is that one cubic foot of water weighs about 1000 oz.

(vi) It is strongly recommended that actual blocks are used to illustrate inch cubes; these greatly add to the reality of the discussion and can be made by sawing up wood of square cross-section, alternatively the pupils can make cubes in connection with geometry and these can be stored for use in arithmetic. (For an interesting method of constructing a cube, and other polyhedra, see an article by A. R. Pargeter. Plaited Polyhedra, *The Mathematical Gazette* Vol. XLIII, No. 344, May 1959.) Work on volumes should be linked up with elementary work in geometry on plans and elevations. In particular, the idea of cross-section needs to be carefully brought out.

(vii) The idea that volumes of similar solids are compared by the cubes of their linear dimensions follows by analogy with areas (see p. 47) by replacing the squared grid by a formation of cubes. Again, the comparison of the surfaces of similar solids by the squares of their linear dimensions is implicit in what was said on p. 47. The importance of these two ideas taken together is capable of vivid practical illustration, particularly in the field of biology. (See *Teaching mathematics in secondary schools*, Ministry of Education Pamphlet 36, p. 114–116.)

Circles and cylinders. (i) The circumference of a circle. The idea of the length of a curve is not a difficult one for the pupil who is used to the measuring tape. Body measurements of chest expansion, waist, etc., are common enough and the natural way for a child to measure the circumference of a circle is to wrap a tape or some cotton round it. The first point to make is that the circumference is proportional to the radius. The pupils should be brought to see that one circle is an enlargement of another. The circumference of circles of different radii may be estimated by stepping off in small steps with compasses, by winding a piece of cotton round a cylinder a number of times (ten is a good number) and by measuring the distance rolled by a wheel in a number of revolutions. When a number of circumferences have been measured, the results can be compared on a graph of circumference against radius, or the ratios may be found and compared.

It needs to be stressed from the beginning that π is the ratio of the circumference of a circle to its diameter. Its value is 3·141 592 65 ..., and there are a number of convenient approximations: $3\frac{1}{7}$ is very slightly nearer than 3·14, but not as near as 3·142. The first use of π is an excellent opportunity to bring in a little of the history of mathematics.

(ii) Any approach to the area of a circle should begin with estimating the areas of circles of different radii by counting squares. (See p. 46.) With less able pupils the formula for the area will be found from these data. But with abler pupils the formula for the area may be deduced from the formula for the circumference as follows.

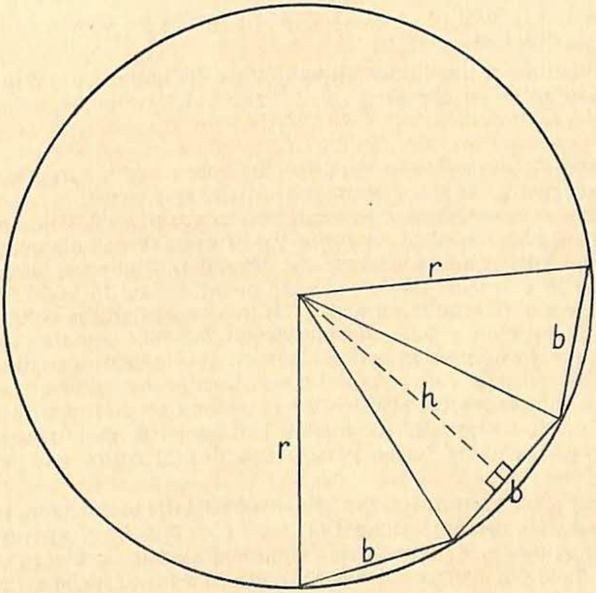


FIG. 9.3

A regular polygon of n sides is inscribed in a circle of radius r and the centre of the circle is joined to the vertices of the polygon (Fig. 9.3). The polygon is thus divided into n congruent isosceles triangles whose bases are the sides of the polygon. Let the sides of the polygon be of length b and their distance from the centre be h. Then the area of each triangle is $\frac{1}{2}bh$ and so the area of the polygon is $\frac{1}{2}nbh$, that is,

$$\text{the area of the polygon} = \tfrac{1}{2} \times (\text{perimeter of polygon}) \times h. \qquad (1)$$

Now note that, as the number of sides of the polygon is increased indefinitely,

(*a*) the area of the polygon approaches as close as we like to the area of the circle,
(*b*) the perimeter of the polygon approaches as close as we like to the circumference of the circle,
(*c*) h approaches as close as we like to r.

Thus the equation (1) gives us

$$\begin{aligned}\text{the area of the circle} &= \tfrac{1}{2} \times (\text{circumference of circle}) \times r,\\ &= \tfrac{1}{2} \times 2\pi r \times r = \pi r^2.\end{aligned}$$

At this stage of the course, the word 'limit' will not generally be used and the above argument will be strange to the pupils. It is thought better by some teachers to consider, as well, a regular polygon of n sides circumscribing the circle. In the same way it is found that the area of the circumscribing polygon approaches as close as we like to πr^2, so that the area of the circle lies between two areas which both approach as close as we like to πr^2.

(iii) The area of the curved surface of a cylinder is easily obtained once the formula for the circumference of a circle is known. If paper round a cylinder is unwrapped to form a rectangle, the pupils should be able to work out the formula for themselves.

(iv) The volume of a cylinder presents little difficulty if the pupils have a clear understanding of the area of a circle and the volume of a solid of uniform cross-section.

Pyramids and Cones. The formula for the volume of a pyramid is not an easy one to derive at the stage where it is usually first met. It may be assumed that pyramids on equal bases with equal heights are equal in volume and then a triangular prism is dissected, as in Fig. 9.4. If a cutter is available, cheese is a good material with which to illustrate the dissection. Many teachers, however, would prefer to construct models of card or paper and to build up a prism from three pyramids. Another approach is to assume that the volume is some fraction of the product of base area and height, in which case the constant $\frac{1}{3}$ is found most easily by considering the six pyramids obtained when the diagonals of a cube are drawn. A third method is to compare the volume of a pyramid and a prism with the same base and height, filling models of stiff cardboard with sand. A convenient size for models to be used is about three inches in height: they can be made during an earlier part of the course and kept for use at this stage.

For the cone, the volume follows by considering the limit (though this word will not usually be used at this stage) as the number of sides of a pyramid's base increases indefinitely. The area of the curved surface is better treated by considering how the surface may be laid out on a plane surface, becoming a sector of a circle.

Spheres. There is no harm in introducing the formulae for the surface area and volume of a sphere without justification, the proofs being deferred until the calculus is reached. The fundamental result connecting a zone of a sphere with the corresponding zone of the circumscribing cylinder may be given to

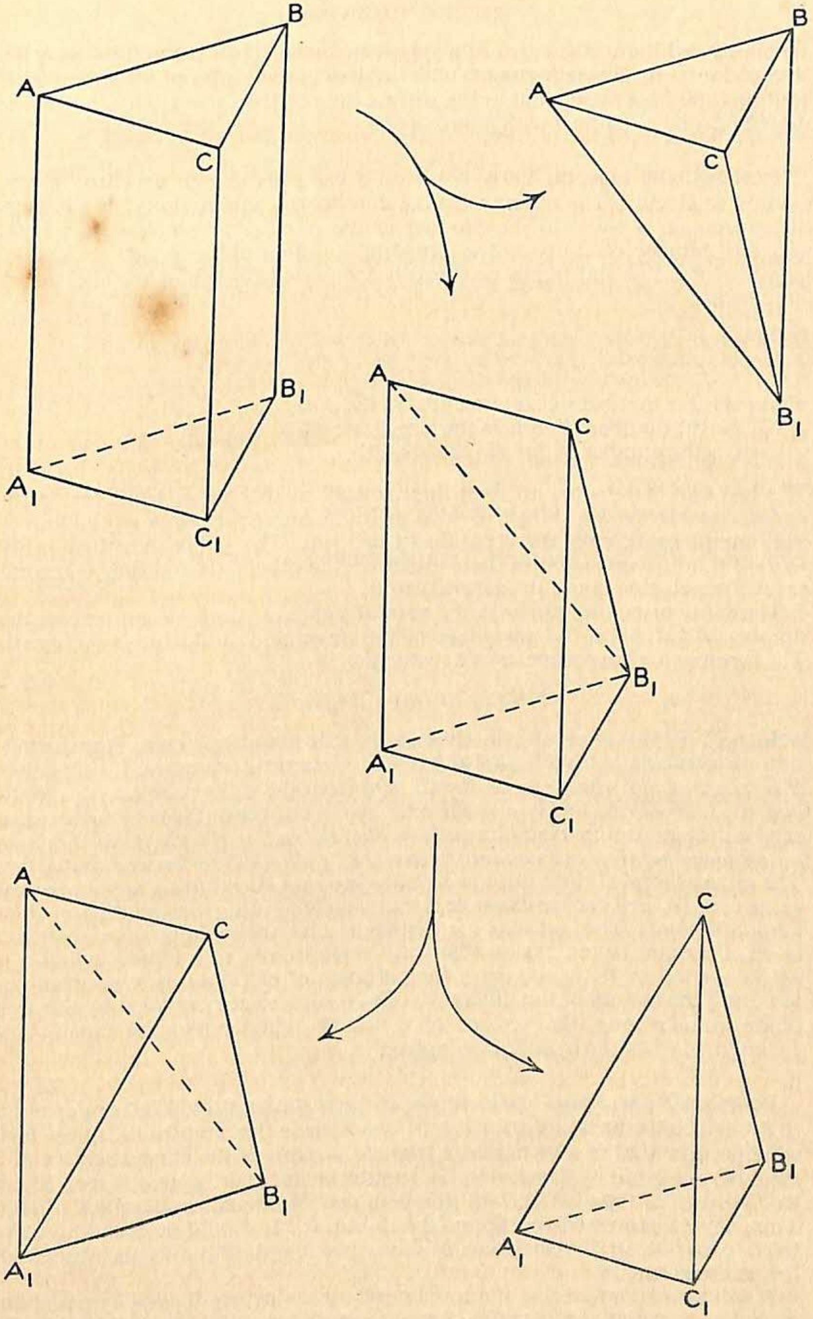


Fig. 9.4

the pupils and the surface area of a sphere deduced. Then the volume may be obtained by considering pyramids with vertices at the centre of the sphere and limiting total base area equal to the surface area of the sphere. This, however, should only be attempted with good classes.

Relations with algebra. Early work on areas provides an excellent introduction to algebra. The statement, 'The number of "square units" in the area of a rectangle is equal to the product of the number of units in the length and the number of units in the breadth,' is often abbreviated to 'Area = length × breadth' and pupils can readily see the advantage of the shorthand statements,

$$A=lb, \qquad l=\frac{A}{b}, \qquad b=\frac{A}{l},$$

where A is the number of 'square units' in the area,
l is the number of units in the length,
b is the number of units in the breadth.

The last two statements are best approached, at this stage, from first principles, considering a rectangle divided up into a number of squares and knowing how many squares there are down one side. The pupils can then easily work out how many squares there are down the other side and they will soon reach a conclusion about the general result.

The same principles apply in the case of volumes. With an increase in the number of variables, the advantage of the shorthand of algebra is increased. The formula for the perimeter of a rectangle,

$$P=2(l+b)=2l+2b,$$

written in these two forms, can give us the rule about brackets. Transformation of formulae is usually first done with numerical examples, for instance, *If* $S=2\pi rh$, *find* r *when* $S=25$, $h=10$, and from there the problem is generalised to, *If* $S=2\pi rh$, *find* r *in terms of* S *and* h.The advantages of factorising can be brought out in examples such as *Find the radius of a circle equal in area to the space between two concentric circles of radii eight inches and ten inches.* The same example brings out the advantage of not substituting any numerical value for π. A well chosen example is more likely to bring this point home than admonishments. The fact that π is written as a letter should help its treatment as an algebraic factor. Again, the total surface area of a closed cylinder is better treated in factorised form for purposes of calculations. Opportunities for using the factors of the difference of two squares are provided by the area of an annular ring, the volume of a hollow cylinder and the theorem of Pythagoras when the hypotenuse is given.

Units and Dimensions. The formulae of mensuration provide an opportunity to introduce the idea of dimensions. If we wish to find an area in square feet, whether it be that of a rectangle, a triangle, a circle or the curved surface of a cylinder, we begin by expressing the lengths in feet and the area is then found by multiplying together certain numbers two of which are numbers of feet. This may be expressed shortly as 'ft × ft = sq. ft'. It should be noted however that we do not, strictly speaking, multiply two lengths but only the number of feet in those lengths.

It will also be noted that if a number of cubic yards is divided by a number of yards the result is a number of square yards, and this observation may be expressed in the symbolic form

$$\frac{\text{cu. yd}}{\text{yd}} = \text{sq. yd.}$$

In the same way, in finding the time taken for a journey, we may divide the number of miles by the number of miles per hour, thus obtaining the number of hours taken. The 'dimensions check' for this is

$$\text{mi.} \div \frac{\text{mi.}}{\text{h}} = \text{h.}$$

Such ideas may be used as a check on the processes used in solving problems. The invaluable habit of thinking in terms of dimensions is at the same time instilled.

Consider, for example, the problem *Find the depth if a cubic foot of sand is spread over a rectangular area* 3 *yd long and* 30 *in. wide*. Some teachers would write the initial stages of the solution along the following lines:

$$\text{Length} = 9\text{ft.} \quad \text{Width} = \frac{30}{12}\,\text{ft.}$$

$$\text{Depth} = \frac{\text{volume}}{\text{length} \times \text{breadth}} = \frac{1}{9 \times \frac{30}{12}}\ \text{ft etc.}$$

Others like to make use of the check provided by dimensions and would write something like:

$$\text{Depth} = \frac{\text{volume}}{\text{length} \times \text{breadth}} = \frac{1}{9 \times \frac{30}{12}}\ \frac{\text{cu. ft}}{\text{ft} \times \text{ft}},$$

$$= \frac{12}{9 \times 30}\ \text{ft, etc.}$$

Conversion to convenient units should be done in preliminary steps, as in the above example. More elaborate problems on change of units, such as

Find in francs per litre the price equivalent to 5s *per gallon if* £1 = 13·8 *fr and* 1 *litre* = 0·220 *gal*

may be treated as follows:

1 gal costs 5s.
(1 litre = 0·220 gal)
1 litre costs 0·220 × 5s.

$$(1\text{s.} = \frac{13\cdot 8}{20}\ \text{fr})$$

$$1\ \text{litre costs}\ 0\cdot 220 \times 5 \times \frac{13\cdot 8}{20}\ \text{fr,} \quad \text{etc.}$$

Later on, some teachers may like to introduce the more compact form:

$$\text{Price} = \frac{5\text{s.}}{1\ \text{gal}} = \frac{5 \times \frac{13\cdot 8}{20}\ \text{fr}}{\frac{1}{0\cdot 220}\ \text{l.}}, \quad \text{etc.}$$

CHAPTER 10

GRAPHS

GRAPHS enter into every branch of school mathematics, as well as into such other subjects as physics and geography. For this reason they should not be regarded as a separate subject to be taught exhaustively at a certain stage, but should be brought in from time to time as occasion arises.

The purpose of any diagram is to convey information in a vivid manner. That of a graph is to exhibit the dependence of one variable on another. The teacher's aim, in graph work, should be to build up this idea of functional dependence, looking forward more to the calculus than to coordinate geometry.

A beginning may be made by examining ready-made diagrams such as can be found in newspapers, particularly in the advertisement columns. These will include pie-diagrams, isotypes and histograms, as well as ordinary graphs, and the observations made from them will at first be qualitative rather than quantitative. There is much to be said for the slogan 'Reading before writing; qualitative before quantitative'. The self-recording barometer provides a good example and rough graphs can be drawn on the blackboard to represent such things as speed of a motor-car, petrol-gauge readings, weight of a boy, rate of consumption of electricity during the course of a day. Pupils should be able to describe in simple terms what a graph tells them and may be encouraged to use their imagination in accounting for the ups and downs. They should also be able to sketch a rough graph on plain paper from such general information as 'the temperature rose steadily from 5 a.m. to 10 a.m., then more slowly, reaching a maximum at 12 noon, etc.'

Full understanding, particularly of a quantitative kind, will not come until pupils can draw graphs for themselves, and this should be begun at an early stage. Very simple examples should be chosen at first, based if possible on the pupils' own experience. Graph paper need not be used, but will soon be found to be convenient. Suitable early examples are:

The number of school periods allotted each week to the various subjects.
A pupil's own marks in a series of tests.
The number of members of the class whose birthdays fall in the different
  months of the year.
The height of a bean plant measured daily.
Takings in the school tuck-shop.
Daily temperature readings.

These may be represented by columns (i.e. by a series of 'vertical' lines on the paper) or, more conveniently, by points indicating the tops of the columns. The points (or the tops of the columns) may be joined by straight lines, to make them show up more clearly.

Some of the work described above will probably have been done in the primary school and further suggestions for it will be found in *The teaching of mathematics in primary schools* and *Mathematics in secondary modern schools*.

When more difficult graphs are drawn, a number of details need to be considered. These concern axes, scales, labelling, the question of intermediate points and the technique of drawing. They cannot all be explained at once and at first the teacher must give detailed instructions as the work proceeds. Thus good habits can be instilled, while difficulties are postponed. Occasions for discussion will arise from time to time and the pupils will gradually acquire a fuller understanding and the ability to decide questions of detail for themselves. The following must be kept in mind from the start, at first by the teacher, later by the pupils:

Axes. The independent variable should always be measured across the page. This of course is a pure convention, comparable with writing from left to right, but teachers should insist on a rigid adherence to it. The pupils will then become accustomed to the idea that the quantity whose variation they are exhibiting is measured up the page and, at a later stage, they will begin to appreciate the meaning of the terms 'dependent variable' and 'independent variable'.

Scales. These will at first be chosen by the teacher. It is more important to choose an easy scale than the largest possible one, and the intention to take readings in decimals should be borne in mind. Thus convenient scales are, for example, 1, 2, 5, 10 or 100 units to the inch or, for larger scales, 1 unit to 2, 5 or 10 inches. The zero of a scale should be shown where reasonably practicable. (It may be noted that gross misrepresentation can easily be achieved, as in some advertisements, by the omission of this. On the other hand it is clearly impossible to show zero for temperatures and dates, and undesirable in some other cases where small variations are being studied.) Numbers should be written along the axes, at intervals of an inch or half an inch. Where one inch represents 5 units, or 0·5 of a unit, it is a great help to put marks to show where the sub-divisions come (Fig. 10.1).

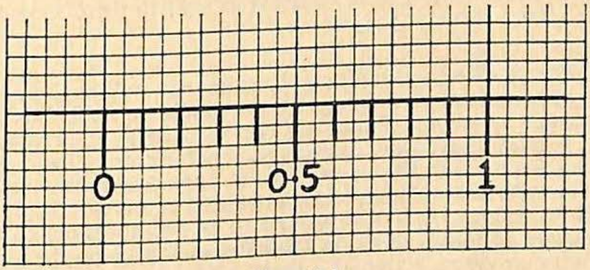


FIG. 10.1

The reading of scales is difficult, as anyone who has used a slide rule knows, and requires careful teaching.

Labelling. Every graph should have a heading, referring primarily to the dependent variable, e.g. 'speed of a train,' 'number of boys in the school from 1939 to 1963,' 'time of sunset through the year.' The axes should be labelled as, for example, 'Time in seconds', 'Number of boys', 'Speed in m.p.h.' This, together with numbers at suitable intervals, is sufficient: it is not necessary to add '1 inch represents 5 m.p.h.'

Where two or more graphs are drawn with the same axes they should be drawn, if possible, in different colours, and should in any case be suitably labelled.

Intermediate points. Examples will be met in which:

(*a*) intermediate points have no meaning;
(*b*) the intermediate points have a meaning, but their position cannot reasonably be guessed;
(*c*) the intermediate points have a meaning and the plotted points show a sufficiently regular distribution to encourage interpolation.

Graphs of type (*a*) are functions of a discrete variable, and include the times of high tide, of sunset and of lighting-up time on successive days, the size of the exterior and interior angles of regular polygons, the result of a series of experiments in which each pupil counts the number of heads in ten throws of a coin or the number of vowels in a line of print (a frequency diagram being drawn to show how many such experiments gave each possible result). In these examples the plotted points will be joined by straight lines to assist the eye in following the variation.

For (*b*), good examples are the number of people in a bus queue, counted at intervals of two minutes; the speed of a motor-car observed every half-minute; the water-level in a tidal river observed at the same time each day. Here again, the points will usually be joined, if at all, by straight lines.

Graphs of type (*c*) should include some where the variation is irregular, but interpolation is reasonably possible, e.g. growth curves, population curves, travel graphs (with uniform velocity assumed, as in the construction of railway time-tables); and some in which the variation follows an algebraical law, e.g. ready reckoners, fahrenheit and centigrade temperatures, squares of numbers, logarithms, simple and compound interest, areas of circles, length of a degree of longitude, distance of the horizon. In these graphs the points will usually be joined by a smooth curve; but, where the points lie approximately on a straight line, or where there is reason to suppose that they ought to do so, it may be more useful to draw a straight line passing as near to them as possible. For this purpose a piece of cotton may be used.

Drawing. The labelling and marking of scales may well be done in ink, but the plotting of points and the joining of them should normally be done with a sharp pencil, HB for choice. Points are best marked by the crossing of two lines, and it is often convenient, more especially for a point inside one of the squares of the graph paper, for the two lines to be parallel to the axes.

The drawing of a smooth curve through plotted points is difficult and much practice is needed. The golden rule is never to go over the same ground twice. To make it easier to follow this rule, the hand should be on the inside of the curve, the wrist or elbow acting as a pivot, and the curve may be drawn in sections, contact being broken off the moment the position becomes uncomfortable or the direction seems to be going wrong.

As the examples mentioned will have indicated, opportunities for drawing graphs occur in many parts of the arithmetic course as well as in other branches of mathematics. In algebra, graphs are of fundamental importance as a means of developing the idea of a function. Gradients, too, can be observed at a very early stage, since quite young pupils appreciate that the steepness of a curve is an indication of rate of change.

Graphs are also drawn in such other subjects as physics, biology and geography and thus provide a connecting link with mathematics and an opportunity for co-operation between teachers in different departments.

It is necessary to add that the drawing of a graph is not an end in itself. Questions should always be asked with a view to bringing out the meaning and purpose of the graph. How does such-and-such a quantity vary? Is the variation regular or irregular? If irregular, can we account for any of the noticeable features? If regular, does it follow an algebraical law?

Interpolation is not the primary purpose of every graph and should not be overworked; but it provides suitable questions in some cases. Travel graphs lead to interesting questions about trains passing each other, and an exponential graph can be used to suggest meanings for fractional indices. But, when all is said and done, a graph is primarily a method of presentation of facts which are already known.

INSTITUTE OF EDUCATION FOR WOMEN
Dept. of Extension
SERVICE
CALCUTTA-17

CHAPTER 11

SQUARES AND SQUARE ROOTS

BEFORE any work is attempted on square root the idea of a square and its associated terminology must be thoroughly understood, e.g. $7 \times 7 = 49$, or the square of $7 = 49$, or $7^2 = 49$.

The study of integers and their squares in the sequences

1	2	3	4	5	...
1	4	9	16	25	...

is useful, and the squares of the early integers should be learnt by heart, possibly as far as 20^2.

The step from one square to the next may be illustrated by diagrams such as

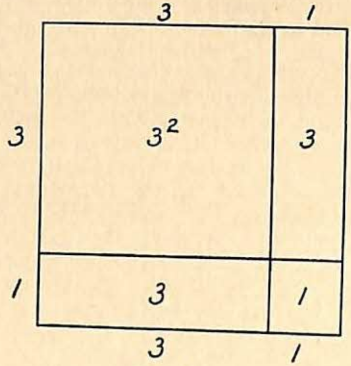


FIG. 11.1

and may be generalised to

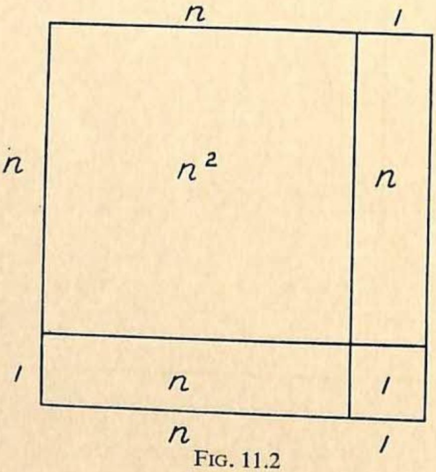


FIG. 11.2

showing the step from n^2 to $(n+1)^2$ by adding n and n and 1, for classes able to take it.

Square roots should be introduced in a situation which shows the need for them. For example, after finding the areas of squares of side 3, 5, 8, ... units it is natural to ask the converse question, 'What is the length of the side when the area is 9, 25, 64, ... sq. units?' This soon raises the problem of finding the side when the area of the square is, say, 7 sq. units. Here is a real challenge and the pupils should first be allowed to make their own crude attempts at meeting it. The table of numbers and squares shows that the side lies between 2 and 3.

Try 2·5; $2{\cdot}5^2 = 6{\cdot}25$—too small.
$2{\cdot}6^2 = 6{\cdot}76$—still too small.
$2{\cdot}7^2 = 7{\cdot}29$—too big.

So $\sqrt{7}$ lies between 2·6 and 2·7, and trial can be continued with 2·65, etc.

The graph of squares should be plotted and used for the estimation of the square roots of numbers, both integral and non-integral, which are not necessarily perfect squares. Applied to the problem of finding $\sqrt{7}$ it might give 2·65. It is easy to see that to find a better approximation, we want a drawing

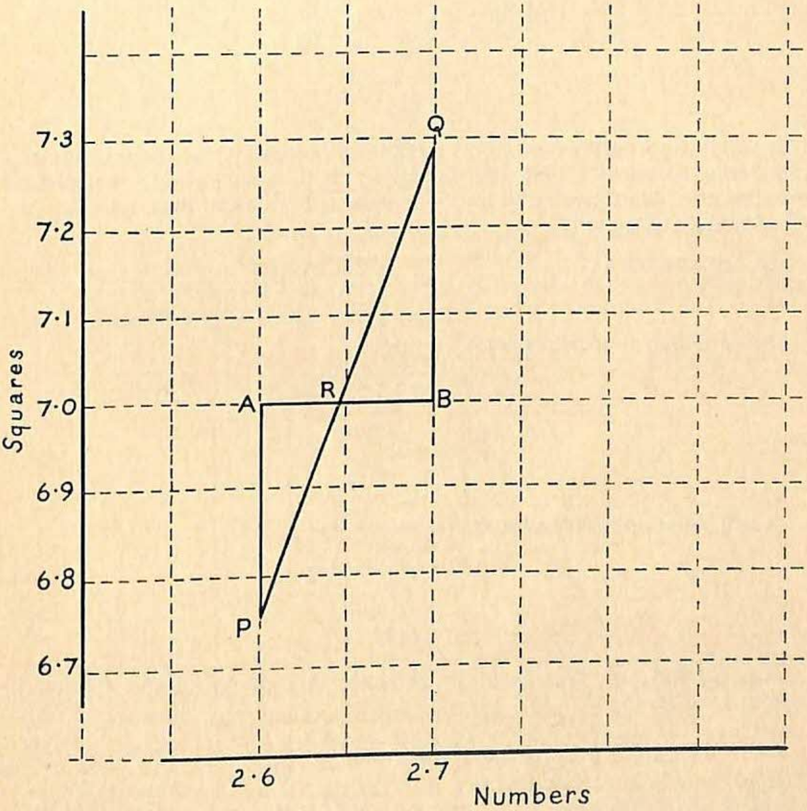


FIG. 11.3

of the graph in this neighbourhood on a larger scale. We have already seen that $2{\cdot}6^2$ is too small by 24 hundredths and $2{\cdot}7^2$ is too large by 29 hundredths. In Fig. 11.3 *AB* is drawn to represent the level of 7 on the squares axis and its length is made 10 units of the graph paper. Let *A* and *B* be above 2·6 and 2·7 on the number axis. *AP*, *BQ* are drawn to represent 24 and 29 on any convenient scale. The straight line joining *P* and *Q* represents the graph closely in this neighbourhood. It meets *AB* at *R* which is estimated to be above 2·64(6) on the number axis. This is a second approximation to $\surd 7$.

The method of finding a square root by factors should not be neglected. It is useful when the number whose square root is to be found is a perfect square and also when there is a single unpaired factor whose square root is known, such as $\surd 2$ or $\surd 3$;

$$\begin{aligned}\text{e.g. (i) } \surd 7056 &= \surd(7\times 1008),\\ &= \surd(7\times 7\times 144),\\ &= \surd\{(7\times 12)\times(7\times 12)\},\\ &= 7\times 12,\\ &= 84.\end{aligned}$$

$$\begin{aligned}\text{(ii) } \surd 98 &= \surd(2\times 7\times 7),\\ &= 7\surd 2,\\ &= 7\times 1{\cdot}414,\\ &= 9{\cdot}898.\end{aligned}$$

A simple approach to Newton's method of estimating the square root of a number is to guess a first approximation to its value, divide the guessed number into the original, and then average the divisor and the quotient to get a better approximation;

e.g. suppose $\surd 2 = 1{\cdot}5$,

$$\text{then a better approximation is } \frac{1{\cdot}5+\frac{2}{1{\cdot}5}}{2},\quad = \frac{1{\cdot}5+1{\cdot}33\ldots}{2},\quad \simeq 1{\cdot}42.$$

$$\text{A still better approximation is } \frac{1{\cdot}42+\frac{2}{1{\cdot}42}}{2},\quad = \frac{1{\cdot}42+1{\cdot}4084\ldots}{2},\quad \simeq 1{\cdot}4142.$$

Alternatively, we can show the extraordinary power of the method by working as follows:

(*a*) work the first stage in fractions, obtaining $\frac{17}{12}$;

(*b*) then the second stage becomes $\frac{1}{2}\left(\frac{17}{12}+\frac{24}{17}\right)$, and working to six places of decimals, say,

$$\text{we have } \tfrac{1}{2}(1{\cdot}416\ 666\ \ldots + 1{\cdot}411\ 764\ \ldots),$$
$$= \tfrac{1}{2}(2{\cdot}828\ 430\ \ldots),$$
$$= 1{\cdot}414\ 215.$$

This may now be compared with the value of $\sqrt{2}$, correct to seven significant figures, 1·414 214.

The pupils may investigate the accuracy of the approximation at any stage by squaring or by carrying the process a stage further.

This is a method for which desk calculating machines are particularly suited.

Fig. 11.2 above suggests yet another method. The difference between $(n+1)^2$ and n^2 is a little more than twice n; so, for example, $40^2 = 1600$ and $41^2 = 1681$; the difference is 81, a little more than twice 40, and $81 \div (2 \times 40)$ gives the needed 1. See Fig. 11.4:

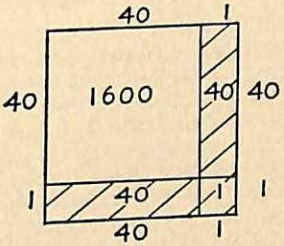


FIG. 11.4

Similarly, suppose $\sqrt{529}$ is required; this is clearly between 20 and 30. The second digit can be obtained by taking $529 - 20^2$, viz. 129, and dividing this by twice 20; $129 \div 40$ gives 3 as the start for the quotient, and 3 is the next figure of the square root; in fact $\sqrt{529} = 23$ exactly. See Fig. 11.5:

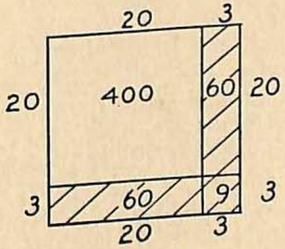


FIG. 11.5

Again, to find the square root of 1984, which lies between 40 and 50, take 40^2 from 1984, leaving 384; divide 384 by twice 40, giving 4 and a remainder, showing that 1984 lies between 44^2 and 45^2; take 44^2 (=1936) from 1984, and divide the remainder, 48, by twice 44, giving 0·5; then $\sqrt{1984} \simeq 44{\cdot}5$, and so on to any desired degree of accuracy.

It should be shown that the square of any one-digit number lies between 1 and 100, i.e. it has 1 or 2 digits; the square of any two-digit number lies between 100 and 10 000, i.e. it has 3 or 4 digits. So, reversing the process, we can show that if a number has one or two digits, its square root has one; if it has three or four, its square root has two, and so on. In general, then, if we mark off the digits in pairs from the decimal point, corresponding to each compartment there will be one digit in the square root.

E.g. of 4 | 43 | ·94 | 49 the square root is 2 | 1 | ·0 | 7
and of 4 | ·43 | 94 | 49 the square root is 2 | ·1 | 0 | 7

It should be noted also that one sequence of figures in the square will provide two different sequences of figures in the square root according to the position of the decimal point; thus 1000, 100·0, 10·00, 1·000 give as square roots 31·62, 10, 3·162, 1. This should be carefully explained and thoroughly understood when first using tables.

These facts will explain the first step in the general method. It will enable certain square roots (e.g. $\sqrt{0{\cdot}01}$, $\sqrt{0{\cdot}0064}$, $\sqrt{1{\cdot}44}$) to be written down, and is also useful in squaring mentally, e.g. 0·1 or 0·07, using the original argument that the number of decimal places in the square will be twice the number in the square root.

Considering the widespread use of tables and mechanical aids to computations nowadays, it may well be argued that there is no need for this somewhat cumbersome and lengthy process to be undertaken at all; but the actual achievement of the desired result by one's own effort undoubtedly gives real enjoyment to some and satisfies the curiosity of many, though they may never do it again. Moreover, it is an elementary example of the increasingly important method of iteration, although it is not actually one used by computers.

The general method should not be attempted before the pupils have enough algebra to understand the theory. It is suggested that an arrangement of the work might be as follows, and the first examples should come out exactly.

To find the square root of 1011·24:

```
            3    1  ·  8
      3 | 10 | 11 | ·24 |
        |  9 |    |     |
     61 |  1 | 11 |     | ............................... (i)
        |    | 61 |     |
    628 |    | 50 |  24 | ...............................(ii)
        |    | 50 |  24 |
```

It is worthwhile to point out that at line (i) in the above work, the 11 124 is $101\,124-300^2$, that at line (ii) the 5024 is $101\,124-310^2$, and so on.

To find the square root of 1984, correct to 2 decimal places:

```
          4    4  ·  5    4   (2)
      4 | 19 | 84 | ·00 | 00 | 00
        | 16 |    |     |    |
     84 |  3 | 84 |     |    |
        |  3 | 36 |     |    |
    885 |    | 48 |  00 |    |
        |    | 44 |  25 |    |
   8904 |    |  3 |  75 | 00 |
        |    |  3 |  56 | 16 |
  89082 |    |    |  18 | 84 | 00
        |    |    |  17 | 81 | 64     ∴ √1984 = 44·54.
```

The process of dividing by twice the part of the root already obtained has been suggested by the diagrams given above. We may also illustrate the process by showing $(a+b)^2$ divided into 4 parts and using $(a+b)^2-a^2=2ab+b^2$, while $\dfrac{2ab+b^2}{2a}=b+$(a fraction), if b is sufficiently small.

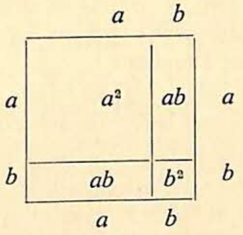


FIG. 11.6

Special attention should be given to finding square roots of decimals with one significant figure, e.g. $\sqrt{0\cdot 1}$, $\sqrt{0\cdot 6}$, $\sqrt{0\cdot 06}$, $\sqrt{0\cdot 0005}$.

When fractions are considered several methods may be employed. For example:

$\sqrt{1\frac{9}{16}}=\sqrt{\frac{25}{16}}=\frac{5}{4}=1\frac{1}{4}$;
$\sqrt{1\frac{1}{2}}=\sqrt{1\cdot 5}=1\cdot 225\ldots$ by tables or a suitable method of approximation;
$\sqrt{1\frac{1}{2}}=\sqrt{\frac{3}{2}}=\frac{1}{2}\sqrt{6}$, etc.

Their relative advantages can be discussed and pupils should be encouraged to make their own choice.

Full understanding of the notation often takes a long time. This can be seen, for example, in such clumsiness as evaluating $(\sqrt{2})^2$ by squaring $1\cdot 414$. It is particularly important throughout work in square roots to test the results frequently by squaring. This can help to make clear such relations as

$$\sqrt{0\cdot 04}=0\cdot 2, \quad \sqrt{15}=\sqrt{3}\times\sqrt{5}, \quad \sqrt{12}=2\sqrt{3}, \quad \sqrt{10}>3.$$

Squaring is also useful in correcting errors, e.g.

$$\sqrt{(9+4)}\neq 3+2 \quad \text{because} \quad 9+4\neq 25.$$

Applications of Pythagoras' theorem to plane and solid figures make useful exercises, and it may come as a surprise to a pupil to find that a line $\sqrt{2}$ inches long can be drawn on paper but cannot be measured exactly with a ruler. It is useful to emphasise that the statement '$\sqrt{2}=1\cdot 414\ldots$' is a way of saying that '$\sqrt{2}$ lies between $1\cdot 4135$ and $1\cdot 4145$.' The idea of 'betweenness' deserves more attention than it commonly receives (see Chapter 13), and it may well be mentioned that $\sqrt{2}$ is a new kind of number, an irrational one, possibly the first such number that the pupil has met.

Many who teach Pythagoras' theorem early would introduce this work on square roots at about the same time in order to solve problems arising from the theorem. It is also needed in the solution of quadratic equations, and in the application of certain formulae. An example is the formula $\sqrt{\dfrac{3h}{2}}$ for the distance in miles of the visible horizon from a height h feet above sea-level, which leads to such questions as to whether Ireland can be seen from the top of Snowdon or Scafell.

Pupils who have learned to factorise the difference of two squares in algebra may be interested if they are shown how a continued fraction can be used to estimate a square root,

e.g.

$$\begin{aligned} &\text{if } x^2=2,\\ &x^2-1=1,\\ &(x-1)(x+1)=1,\\ \text{so}\quad &x-1=\frac{1}{1+x}\\ \text{and}\quad &x=1+\frac{1}{1+x}=1+\cfrac{1}{1+1+\cfrac{1}{1+x}}\\ &=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}}} \end{aligned}$$

The practical application of this idea consists in making an approximation by terminating the continued fraction at some stage. The pupil may be led to discover experimentally that the more terms that are retained, the better the approximation. Denoting successive approximations by $a_1, a_2, a_3, \dots$ we have

$$a_1=1+\tfrac{1}{2}= \qquad 1{\cdot}5.$$

$$a_2=1+\frac{1}{2+\frac{1}{2}}=1+\frac{2}{5}= \qquad 1{\cdot}4.$$

$$a_3=1+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}} \quad =1+\cfrac{1}{2+\cfrac{2}{5}}=1+\frac{5}{12}=1{\cdot}42 \text{ approx.}$$

$$a_4=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}}} \quad =1+\cfrac{1}{2+\cfrac{5}{12}}=1+\frac{12}{29}=1{\cdot}414 \text{ approx.}$$

Tables. To return to the argument that the freer use of logarithm and square root and other tables makes any attention to the long methods here described unnecessary: individual teachers must decide this in terms of their own scale of values and the circumstances in which they work. Ultimately one hopes that pupils will have as near a universal view of mathematics as possible, and that anything they learn will be intrinsically worth knowing. Even if square root tables are accepted and used, the question of how the tables are compiled might reasonably be asked, and pupils could usefully construct a line or two. They may be shown both forms of the tables, and reminded that a table of squares used in reverse will provide square roots. And even when they calculate square roots using logarithms, they may be informed that Henry Briggs calculated the first set of logarithm tables in England by truly heroic feats of the square root process. In any case, ideally, a pupil should have at his disposal as many tools as he can reasonably master at any stage of his work.

CHAPTER 12

LOGARITHMS

PROFESSOR JOHN PERRY in his 'lectures to working men' said, 'Some friends of mine assert that no man or boy ought to be allowed to use logarithms until he knows how to calculate them. . . . This is a hard saying. It is exactly like saying that a boy must not wear a watch or a pair of trousers until he is able to make a watch or a pair of trousers.' Many teachers today would go further and say that it is quite practicable to learn to use a table of logarithms as a tool of calculation before understanding how the process works or how it came to be invented. In this approach the slide rule makes a convenient starting point especially if home-made models are constructed from strips of logarithmic graph paper. The aim should be to acquire through practical experience some understanding of the law which is formulated later in the form $\log ab = \log a + \log b$. The use of the tables is then easily developed.

Many children in grammar schools, however, will derive greater satisfaction from an introduction through indices, because this gives some understanding of underlying principles from the outset. It is not necessary to have much previous knowledge of negative and fractional indices. Negative integral indices can be introduced by inspecting the table

$$\begin{array}{cccc} \ldots & 10^3 & 10^2 & 10^1 \\ \ldots & 1000 & 100 & 10 \end{array}$$

and noting that dividing by 10 in the second line is equivalent to subtracting 1 from the corresponding index in the top line. A continuation of this process leads to

$$\begin{array}{ccc} 10^0 & 10^{-1} & 10^{-2} \ldots \\ 1 & 0{\cdot}1 & 0{\cdot}01 \ldots \end{array}$$

Some practice with positive and negative indices can be given by expressing large and small numbers in standard form and performing simple calculations with them. Standard form will also be useful later on in determining characteristics.

The approach through indices. We can begin with the bald statement that at the beginning of the seventeenth century a Scotsman called John Napier devised a method of multiplying two numbers by adding two other numbers, a much simpler operation. We can then ask if examples of such a process have ever come within our pupils' experience and eventually obtain examples such as $2^3 \times 2^2 = 2^5$. Then a table of powers of 2 from, say, $\frac{1}{8} = 2^{-3}$ to $4096 = 2^{12}$ can be written on the board and used to multiply any two numbers in the table, provided that their product also falls within its range. The problem of how to apply the method to other numbers will soon be raised and can be solved, as a first approximation, by drawing the graph of powers of 2.

In effect this graph introduces such approximations as $7 = 2^{2{\cdot}8}$ and is a temporary substitute for a definition of fractional indices. In the unlikely event

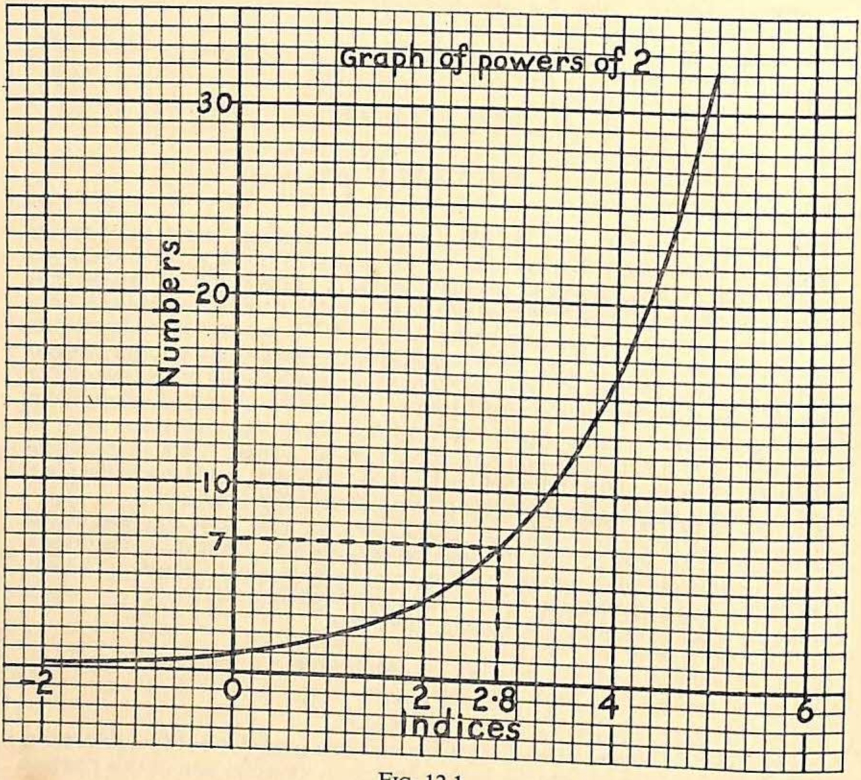


FIG. 12.1

of a pupil asking how 2·8 two's can be multiplied we must explain the need of a fresh definition and examine a few particular cases, e.g.

$2^{\frac{1}{2}} \times 2^{\frac{1}{2}} = 2$, and so we define $2^{\frac{1}{2}}$ to be $\sqrt{2}$,

$2^{\frac{2}{3}} \times 2^{\frac{2}{3}} \times 2^{\frac{2}{3}} = 2^2$, and so we define $2^{\frac{2}{3}}$ to be $\sqrt[3]{2^2}$.

General definitions and manipulative exercises with fractional indices are best postponed, since they distract attention from the main purpose.

Dissatisfaction with the accuracy and range of the graph will soon be expressed, and then the logarithm table can be introduced. We must first point out that powers of 10 are more convenient than powers of 2 and the story of Briggs' journey to Edinburgh to discuss this point with Napier is worth telling here.*

We can demonstrate the advantage of the change of base as soon as the tables can be read. From the graph, $7 = 2^{2 \cdot 8}$ and $70 = 2^{6 \cdot 1}$ approximately; from the tables, $7 = 10^{0 \cdot 85}$ approximately. Hence $70 = 10^1 \times 10^{0 \cdot 85} = 10^{1 \cdot 85}$. If the base 10 is used, the decimal part of the logarithm is not altered by a change in the position of the decimal point of the number, and this greatly reduces the bulkiness of the tables.

Use of the table is conveniently introduced in three stages: (i) for numbers between 1 and 10, (ii) for numbers greater than 10, (iii) for positive numbers less than 1.

---

* F. Cajori, *A history of mathematics*. (Macmillan), p. 150.

Stage (*i*). Some teachers like to use an index method of setting out for some time, others go over to a tabular setting out immediately or after only one or two lessons. The former school believes that reiteration of index statements leads in the end to a better understanding of the nature of logarithms; the latter claims that logarithms should be regarded as a tool and the first consideration is the most efficient use of this tool. Even if the index method is used, pupils should be trained from the start to add the logarithms by writing one underneath the other in the margin.

It will be found useful in preparation for Stage (ii) to allow some answers to exceed 10. For example,

$$\begin{aligned} 6{\cdot}123 \times 3{\cdot}947 &= 10^{0\cdot 7870} \times 10^{0\cdot 5963}, \\ &= 10^{1\cdot 3833}, \\ &= 10^{1} \times 10^{0\cdot 3833}, \\ &= 10 \times 2{\cdot}42, \\ &= 24{\cdot}2. \end{aligned} \qquad \begin{array}{r} 0{\cdot}7870 \\ 0{\cdot}5963 \\ \hline 1{\cdot}3833 \end{array}$$

Stage (*ii*). A few examples like $2{\cdot}8 = 10^{0\cdot 4472}$, $28 = 10 \times 2{\cdot}8 = 10^{1\cdot 4472}$, $280 = 10^{2} \times 2{\cdot}8 = 10^{2\cdot 4472}$ etc. together with an examination of the table $10 = 10^{1}$, $100 = 10^{2}$, $1000 = 10^{3}$, quickly lead to the following table of characteristics:

Range of numbers	1–10	10–100	100–1000
Characteristic	0	1	2

This approach may usefully be supported by sketching the graph of powers of ten on the blackboard. This rubs in the interpolation based on the above table and shows in the most striking way the relationship between numbers and their logarithms. Difficulties of scale and the limited number of points which it is possible to plot (without begging the question by using the table) make this an unsuitable exercise for the pupils unaided.

It may also be pointed out that the characteristic is easily found if the number is expressed in standard form. For example if 23 800 is written in the form $2{\cdot}38 \times 10^{4}$, it is immediately clear that the characteristic is 4.

Stage (*iii*). This can be introduced (not necessarily immediately after Stage (ii)) by going from $2{\cdot}8 = 10^{0\cdot 4472}$ to $0{\cdot}28 = 10^{-1} \times 2{\cdot}8 = 10^{-1+0\cdot 4472}$. The pupil expects this index to be written as an ordinary negative number $-0{\cdot}5528$ and the disadvantage of doing this can be brought home forcibly by setting pairs of exercises such as: Find the number whose logarithm is (i) $-0{\cdot}5528$, (ii) $-1 + 0{\cdot}4472$. As soon as the need to write numbers in this unusual form is appreciated some teachers introduce the bar notation. Others pass through a transition stage in which the characteristic and mantissa are written separately. For example,

$$\begin{aligned} 0{\cdot}0048 \times 0{\cdot}58 &= 10^{-3} \times 4{\cdot}8 \times 10^{-1} \times 5{\cdot}8 \\ &= 0{\cdot}002\ 784 \text{ approx.} \end{aligned}$$

$$\begin{array}{l|l} 0{\cdot}0048 & -3 + \ {\cdot}6812 \\ 0{\cdot}58 & -1 + \ {\cdot}7634 \\ \cline{2-2} & -4 + 1{\cdot}4446 \\ 0{\cdot}002784 & -3 + \ {\cdot}4446 \end{array}$$

This separation is particularly useful when a negative characteristic has to be subtracted, a case that often proves troublesome at first.

$$\frac{0{\cdot}875}{0{\cdot}957}=0{\cdot}914 \text{ to 3 sig. fig.}$$

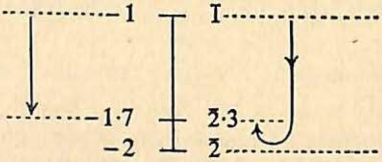
$$\begin{array}{l|l} 0{\cdot}875 & -1+{\cdot}9420 \\ 0{\cdot}957 & -1+{\cdot}9809 \\ \hline 0{\cdot}9143 & -1+{\cdot}9611 \end{array}$$

The working is different according to the method of subtraction used, e.g.

$$\begin{array}{r} -2+1{\cdot}9420 \\ -1+\ \ {\cdot}9809 \\ \hline -1+\ \ {\cdot}9611 \end{array} \qquad \text{or} \qquad \begin{array}{r} -1+1{\cdot}9420 \\ 0+\ \ {\cdot}9809 \\ \hline -1+\ \ {\cdot}9611 \end{array}$$

(It may be pointed out that this calculation is performed more easily by writing the fraction in the equivalent form $\frac{8{\cdot}75}{9{\cdot}57}$.)

The bar notation should be seen as a convenient but not indispensable part of logarithmic calculation. At first some practice with equivalences like $-1{\cdot}7=\bar{2}{\cdot}3$ is desirable and a diagram may be useful.



Much practice with exercises such as $\bar{2}+\bar{3}$, $\bar{3}-\bar{2}$, $\bar{2}-\bar{3}$, $\bar{3}{\cdot}3-\bar{3}{\cdot}2$, $\bar{3}{\cdot}2-\bar{2}{\cdot}3$, etc. will be necessary. It must be remembered that two or more methods of subtraction may be in use in the class. For instance $\bar{2}{\cdot}5-\bar{3}{\cdot}7=0{\cdot}8$ may be worked by saying '7 from 15 is 8, $\bar{3}$ from $\bar{3}$ is 0;' by saying '7 from 15 is 8, $\bar{2}$ from $\bar{2}$ is 0;' or by saying '8 must be added to 7 to make 15, add 1 to $\bar{3}$—this makes $\bar{2}$, 0 must be added to $\bar{2}$ to make $\bar{2}$.'

The theory of logarithms. This hardly comes within the scope of arithmetic at all, and yet important properties of the logarithm function are self-evident from its graph and an informal discussion of them is probably useful. In any case the ability to draw a quick sketch of $y=\log_{10} x$ is valuable. Some of the properties which may be mentioned at this stage are:

(i) The logarithm of a number greater than 1 is positive; the logarithm of a number between 0 and 1 is negative; the logarithm of a negative number is not defined. $\log 1=0$.

(ii) The logarithm increases steadily as the number increases and therefore no two numbers have the same logarithm. Also every index is the logarithm of one and only one number.

(iii) It follows from (ii) that the logarithm of any number between, say, 10^2 and 10^3 is $2+h$, and of any number between, say, 10^{-2} and 10^{-3}, $-3+k$, where h and k are positive fractions. Hence the graph summarises the process of determining the characteristic. If it is thought necessary to give a rule for finding the characteristic, the simplest is probably one based on the expression of the number in standard form. The characteristic is the displacement from the *units* place of the first (i.e. the left hand) significant figure reckoned positively if it is to the left of the units place, negatively if it is to the right.

Powers and roots of a positive number smaller than 1. When the characteristic is negative, the process of finding a root is sometimes a little troublesome. It may be worth pointing out to teachers that it is convenient to choose for the first calculation of a root the converse of a previous calculation of a power:

(i) *Evaluate* $(0{\cdot}0555)^2$.

0·0555	$\bar{2}{\cdot}7443$
	2
0·00308	$\bar{3}{\cdot}4886$

Details of working

$(-2+{\cdot}7443)\times 2$
$=-4+1{\cdot}4886$
$=\bar{3}{\cdot}4886$

(ii) *Evaluate* $\sqrt{(0{\cdot}00308)}$. We retrace the previous working step by step in the opposite order.

0·00308	$\bar{3}{\cdot}4886$
0·0555	$\bar{2}{\cdot}7443$

Details of working

$\bar{3}{\cdot}4886=-4+1{\cdot}4886$
Half of this is $-2+{\cdot}7443$.

Antilogarithms. There is a division of opinion on the use of antilogarithm tables. Some teachers hold that, as there are no 'anti-trigonometrical tables', one has to learn to use most tables both ways, and it is best to do this with the logarithm table as well. To encourage the use of two tables is only to introduce another possible source of error.

Against this it is argued that it is important to acquire the habit of glancing at the heading of a table before using it. (Some of us only learn this when we have tried to catch a Sunday train on a weekday.) Numerical tables are designed to be read forwards and they are always more troublesome to read backwards. The antilogarithm table saves some of this trouble and is quicker. Teachers who hold this view therefore encourage its use from the beginning.

The setting out of logarithmic calculations. Since the purpose of logarithms is mainly to ease the burden of calculation, the setting out should be as simple and economical as is consistent with order and accuracy. The fundamental principle is that addition or subtraction is likely to be more accurate if it is done 'up and down' rather than 'across' the page.

The logarithmic work should be separated from the statement of the calculation and should not be crammed into a margin. In particular, it is wise to leave plenty of room for developments on the logarithm side of the table. The following examples are offered as general suggestions, though many teachers would probably want to make modifications in detail.

$$\frac{18{\cdot}7\times 15{\cdot}8}{216\times 0{\cdot}047}=29{\cdot}1 \text{ to 3 sig. fig.}$$

18·7	1·2718	
15·8	1·1987	
Numerator	2·4705	2·4705
216	2·3345	
0·047	$\bar{2}{\cdot}6721$	
Denominator	1·0066	1·0066
29·10		1·4639

$$428{\cdot}5\sqrt{\frac{0{\cdot}8512}{0{\cdot}9284}}=410{\cdot}2 \text{ approx.}$$

0·8512	$\bar{1}{\cdot}9300$
0·9284	$\bar{1}{\cdot}9677$
Quotient	$\bar{1}{\cdot}9623$
Square root	$\bar{1}{\cdot}9811$
428·5	2·6319
410·2	2·6130

It is always useful to scrutinise a result for reasonableness. Here the square root is obviously a little less than 1 and the answer must be a little smaller than 428·5.

The calculation of part of the logarithm table. This can generate considerable interest and is well worth doing when some understanding of fractional indices has been reached. There are several methods:

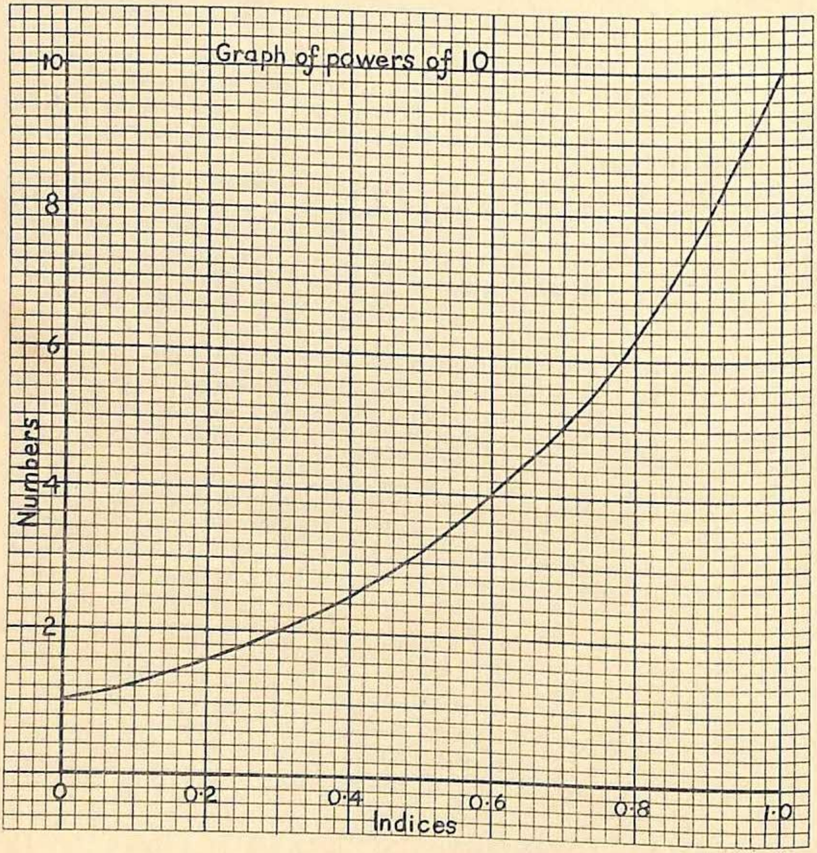


FIG. 12.2

(i) Logarithms can be deduced from powers of a number near 1, e.g. 1·1. This method is fully explained in *The teaching of algebra in schools* on pp. 50–52, and it is particularly suitable if a calculating machine is available.

(ii) Rougher approximations can be obtained from a graph of powers of 10. We illustrate by finding the logarithms of the integers from 1 to 10. By repeated use of square roots we find $10^{\frac{1}{2}}=3{\cdot}162$, $10^{\frac{1}{4}}=1{\cdot}779$, $10^{\frac{1}{8}}=1{\cdot}333$. Then $10^{\frac{3}{8}}=10^{\frac{1}{4}+\frac{1}{8}}=10^{\frac{1}{4}}\times 10^{\frac{1}{8}}=1{\cdot}779\times 1{\cdot}333\simeq 2{\cdot}372$. Approximations to $10^{\frac{5}{8}}$, $10^{\frac{3}{4}}$, $10^{\frac{7}{8}}$ are found similarly and a graph plotted from these values easily gives the logarithms of the integers from 1 to 10 correct to 2 decimal places (Fig. 12.2).

(iii) A refinement of method (ii) yields 4-figure accuracy without too much work and, of course, if this can be secured it is much more impressive and greatly increases interest. Consider first the logarithm of 2. We begin by looking for a power of 2 which can also be expressed approximately as a power of 10; a good one is $2^{10}=1024\simeq 10^3$. We write this as

$$2^{10}=10^3\times 1{\cdot}024,$$

and seek an approximation to 1·024 as a power of 10 by the process of repeated extraction of square roots. Continuing the process given in method (ii) we have

Index of 10	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$	$\frac{1}{256}$
Number	1·154	1·074	1·036	1·018	1·009

In the last three pairs of this table, the indices in the top line are proportional to the decimal parts of the numbers in the bottom line. Thus within the

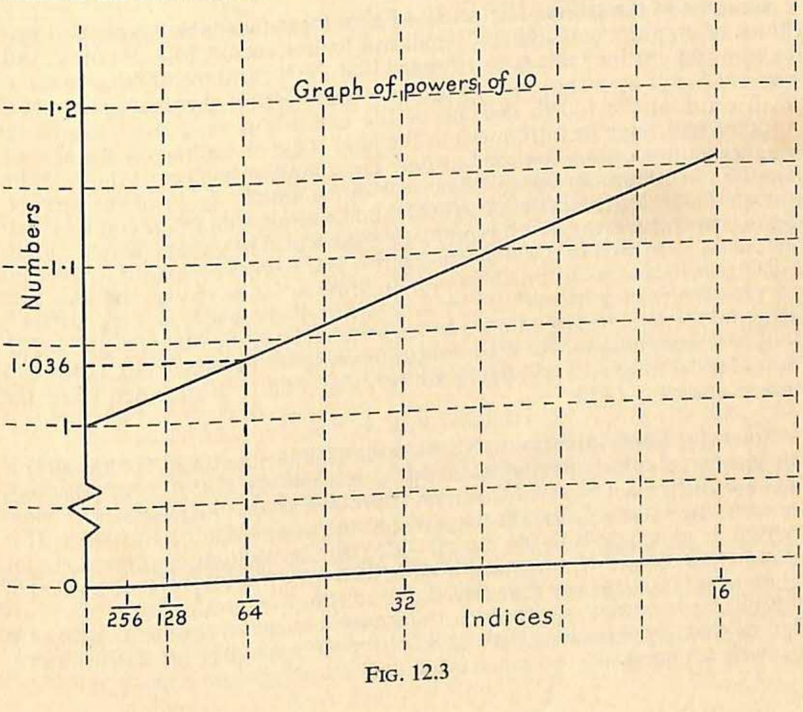


FIG. 12.3

accuracy of our calculations the part of the graph from the number 1 to the number 1·036 is a straight line (Fig. 12.3). Now 1·024 is within this range and, as it happens, 0·024 is two thirds of 0·036. Therefore, the index of 10 which gives 1·024 is $\frac{2}{3} \times \frac{1}{64} = 0{\cdot}0104$, to four decimals.

Therefore $2^{10} = 10^3 \times 10^{0\cdot0104} = 10^{3\cdot0104}$
Therefore $2 = 10^{0\cdot30104}$ which has an error of only 1 in the fifth decimal.

Good practice with indices can be had by calculating the logarithms of the next few primes. (It will, of course, be pointed out that the logarithms of composite numbers do not present a fresh problem.) The value 0·47711 for log 3 can be obtained from $3^4 = 81 = 1{\cdot}0125 \times 80 = 1{\cdot}0125 \times 2^3 \times 10$.
log 5 is calculated as log 10 − log 2.
The value 0·84510 for log 7 can be found from

$$7^4 = 2401 \simeq 1{\cdot}0004 \times 2^3 \times 3 \times 10^2.$$

It is also instructive to obtain the rougher approximation 0·84514 from $7^2 = 49 = 0{\cdot}98 \times 50$.

Finally it should be pointed out that these are elementary methods suitable for the calculation of just a few logarithms. If the question is asked, 'How are the actual tables made?' the short answer is that a number of comparatively widely-spaced basic values are calculated by means of logarithmic series and by using the factors of composite numbers, and intermediate values are found by means of interpolation formulae. Both these stages are facilitated nowadays by the use of calculating machines, but it is noteworthy that in 1717 Abraham Sharp published a table giving, to 61 decimal places, the logarithms of all numbers from 1 to 100 and of prime numbers from 100 to 1097.

Accuracy of the tables. The usual 4-figure logarithm table gives the logarithms of numbers containing 3 significant figures correct to 4 decimals. But for compactness the table is so arranged that the logarithms of 4-figure numbers are found by means of average differences. Hence there is sometimes a small error in the fourth decimal of the logarithms read from the table. Another error may be introduced in the final stage of reading off the answer whether antilogarithms are used or not. The cumulative effect of these errors may be to produce an appreciable error in the fourth figure of the answer, though this seldom exceeds 2. Pupils should be aware that in general this final figure is untrustworthy, but it is a pity to make a firm rule that answers should always be shortened to 3 significant figures. It is sometimes useful to have all four figures given with the abbreviation 'approx.' added as a reminder that the final figure may be out by 1 or 2. Some teachers favour writing the final figure in brackets to show that it is not as reliable as the others, e.g. 27·1(6). It is, of course, possible for the 3-figure answer to be incorrect, and, indeed, to be less accurate than the 4-figure answer, e.g. if 27·16 is obtained when the correct answer is 27·14.

Notes. (1) Some teachers advocate the use of slide rules instead of logarithms for incidental calculations up to about O level of the G.C.E. and point out that much time can be saved this way. An introduction to its use can be made in much the same way as was suggested at the beginning of the chapter. It is helpful to some pupils to use a card slide rule for addition, and then one for multiplying powers of, say, 2, calibrating their own from a graph of powers of 2 before going on to use a manufactured one. (See Fig. 12.4.)

(2) Care is always necessary in using tables to guard against copying the figures wrongly. Mistakes such as 4722 for 4772 or 2837 for 2387 are particularly common.

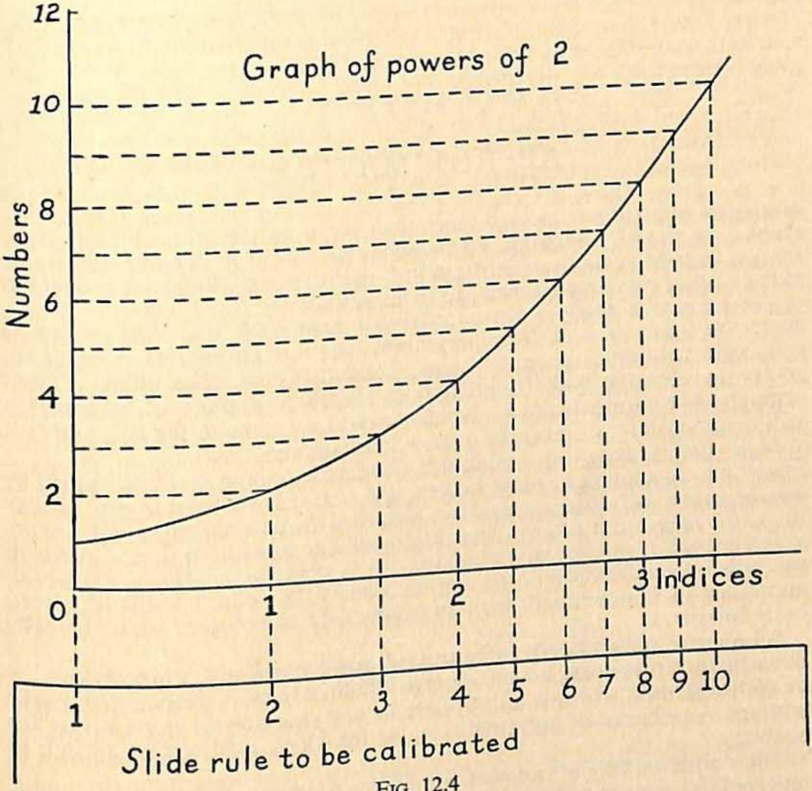


FIG. 12.4

(3) A refinement in the use of differences which can be introduced at a late stage consists in working from the nearer entry in the main table. Thus log 2188 is likely to be found more accurately as well as more quickly by subtracting 0·0004 from log 2190 rather than by adding 0·0016 to log 2180. In any case the habit of checking that the logarithm obtained after applying the difference lies between the adjacent entries in the table is an excellent one, especially in readiness for the use of tables where differences are subtracted.

(4) When looking up antilogarithms greater accuracy can be obtained by using half of the logarithm table and half of the anti-logarithm table, the half with the small differences.

(5) It should be known that 5-, 6- and 7-figure logarithm tables are available when greater accuracy is needed, as for example in navigation. Some teachers start their pupils off on 3-figure logarithms. Some experience of different tables is desirable to put the 4-figure tables in their proper perspective.

(6) A certain amount of practice in the evaluation of formulae by tables of various kinds is needed. For example, in $\frac{a^2+b^2-c^2}{2ab}$, the numerator can conveniently be calculated by tables of squares and the calculation can then be completed by logarithms.

(7) Separate tables of logarithms are much more convenient than tables at the back of a text book and should be supplied whenever possible.

CHAPTER 13

APPROXIMATIONS

IDEAS on approximation tend more than those on the other subjects to be vitiated by slovenly thinking. The superstition that if nothing is perfect one thing is as good as another is not confined to the young. Mathematics ought to make a better contribution than this to human life.

The important thing about approximate answers is that they should be correct as far as they go. If we only know that Jones lives in Hampshire, we can begin to locate him with 'England, South Coast, Hampshire, . . .' but must not continue with 'Bournemouth'; each new piece of information improves our knowledge of his residence until we come to the last, and that will be wrong if, for instance, he lives in Southampton.

Many pupils (and others) think that increased accuracy can be obtained by carrying a calculation to more figures, and that it is virtuous to give greater accuracy than that which is asked for. But when the data are only approximate, the extra figures are often wrong and spoil the answer. If Jones' room in Southampton is marked approximately by a point on a map of the world, successive magnifications of the map may show the point in a district, street, and house in Bournemouth with an appearance of accuracy which is really quite fallacious.

All approximation needs to be treated intelligently and, while we can lay down guiding principles, we cannot give infallible rules. A method or a degree of approximation which is appropriate to one case may be quite wrong for another. An element of judgment is called for which children find difficult to acquire.

The unit in which a measurement is given often indicates the accuracy involved. A distance of 56 miles between two towns is probably given to the nearest mile and the unit chosen is appropriate: it would be ridiculous to give it to the nearest yard. On the other hand the distance of the Sun, 93 000 000 miles, is most appropriately given in miles because we do not use a unit between a mile and the distance of the Sun, but there is no suggestion that the distance is given to the nearest mile.

When something is being measured it has to be decided how to record the result of the measurement. An infant-school child may well be asked to measure the length of a book to the nearest inch or the length of a room to the nearest pace, but the height of the child may be recorded in completed inches and fractions of an inch. Time, too, is measured in different ways: a person will often give the time of day to the nearest five minutes; and age is given to the nearest year or month below the age of a person.

Care should be taken when recording a reading from a calibrated instrument. A clinical thermometer can and should be read to the nearest mark, that is to the nearest 0·2° F. However, although the speedometer of a car may be calibrated to the nearest m.p.h., it is required by law only to be correct within 10% at speeds in excess of 10 m.p.h. and frequently a speedometer fails to register any speed when the vehicle is travelling at a walking speed. On the other hand the medium wave band on a wireless set is often calibrated at every ten metres and adjustment for good reception corresponds to an accuracy of about one metre. A common fault in reading a scale is to write 0·125 (3 sig.

fig. implied) when, in fact, all that can be read off is, 'About half way between 0 and $\frac{1}{4}$.' 0·1 would probably be a better answer.

In mathematics care should always be taken to state how accurately any given measurement has been made and often this will be done by using significant figures or decimal places. These two ways of expressing a degree of accuracy tend to confuse pupils, so it is better to introduce one well before the other. Decimal places are found easier and may first arise in connection with division of fractions. $\frac{1}{3}$=0·3333…, which cannot be written in full, but $\frac{1}{3}$=0·33 correct to the nearest hundredth, or to two places of decimals, and so on.

If pupils have been used to giving an answer correct to so many places of decimals, they wonder what is the point of having a different way of expressing their answer with significant figures. There are two fundamental ideas involved here. The first is that the number of significant figures in a measurement gives an indication of the difficulty involved in making the measurement. Thus, given the appropriate apparatus, it is about equally easy to measure a length to two significant figures whether it be about 5 inches or 50 yards; and three significant figures requires a good deal more care in both cases. To measure a length correct to four significant figures will require special apparatus not generally available in the classroom.

The other fundamental idea involved in significant figures may be brought out by an example like the following. Consider an ideal circle of radius 4 inches. If we take the value of π to 1, 2, …, 6 significant figures, the following results are obtained from calculating the circumference of the circle:

Value for π	No. of sig. fig.	Circumference in inches
3	1	24
3·1	2	24·8
3·14	3	25·12
3·142	4	25·136
3·141 6	5	25·132 8
3·141 59	6	25·132 72

Now observe that, if the length of the circumference were rounded off to the same number of significant figures as the value of π, the answer would be, with one exception, correct to that order of accuracy. This illustrates the idea that significant figures roughly give the order of accuracy which can be carried through a calculation. As a general rule, a result should not be given to a greater number of significant figures than the *least* accurate of the data.* If, for instance, we calculate the area of a rectangle given as 13·6 in. by 2·7 in. the area may be stated as 36·72 sq. in., but if the data were only correct to the nearest tenth of an inch, as is implied by the way in which they are written, the area may be anywhere between

13·55 × 2·65 sq. in. = 35·9075 sq. in.
and 13·65 × 2·75 sq. in. = 37·5375 sq. in.

In this case, where we might expect two-figure accuracy, the second figure is uncertain; here we may write the answer, 36·72 sq. in., as, 37 sq. in. approx.

Significant figures are best introduced with integers rather than decimals to avoid confusion with decimal places. The distance of the moon is a good starting point: a member of the class may be able to give 240 000 miles (two significant figures) and the fact that the distance varies will probably arise in discussion. The average distance between the centres of the Earth and Moon has been given as 238 857 miles (six significant figures), the distance varying from 221 463 miles to 252 710 miles. The question about when a zero is significant will soon crop up and examples from various sources can help to clarify the pupils' ideas: the road distance from Aberdeen to Blackpool is 308 miles, to Truro is 650 miles, and to Swansea is 500 miles (all zeros significant, the distances being measured between selected points in the towns); the distance travelled by light in one second is 186 000 miles (all zeros not significant: 186 271 miles is a better figure); the mean distance of the sun from the earth is 93 004 000 miles (two zeros significant, three not). At a later stage, the idea that zeros are significant only when they have to be retained for any position of the decimal point can be brought out by writing 6·05 m as 0·006 05 km and 6050 mm.

---

* An exception to this is in finding averages.

Pupils should be encouraged to think for themselves how accurate their answers are. Cases will arise in which the best answer that a pupil can get is not expressible in any regularly used form such as 'to the nearest penny', 'to 3 significant figures', or 'to the nearest minute'. If, for instance, half the answer has been found to be £2 4s. 8d. to the nearest penny, the answer may be given as £4 9s. 4d. within a penny. Or if the angles A and B of a triangle are 43° 13′ and 77° 20′ (to the nearest minute), then $C = 59° 27′$ within a minute, but not necessarily to the nearest minute. This should not be the regular form for answers but should be accepted when circumstances prevent the use of the ordinary forms.

The use of tables* often gives pupils a false sense of accuracy. Calculating the hypotenuse of a triangle with two sides 2·8 in. and 4·1 in., the answer 4·965 in. is obtained but it is unreasonable to give more than two significant figures in the answer when the data are known to be given to 2 sig. fig.

Many mistakes in the use of tables can be avoided by having some idea of the magnitude of the value being found, particularly when there is a possibility of having turned to the wrong set of tables by mistake, as with logarithms and anti-logarithms. The decimal point, too, in looking up the square of a number, is best found by an approximation, e.g. $0{\cdot}087^2$ is a little less than $0{\cdot}09^2 = 0{\cdot}0081$. From tables, $0{\cdot}087^2 = 0{\cdot}007\,569$. As with square roots, the approximation is best done *before* consulting the tables.

In everyday life, calculating how much of some material to order, whether it be cloth, wood, paint, lino, fertiliser etc., the precise amount needed for the job seldom matters because in practice it is wise to have a certain amount to spare, and only occasionally are the materials cut to exact sizes for customers. This aspect of calculations tends to be overlooked in many text-books. The need for people to be skilled at precise calculations has been reduced by the increasing use of graduated weighing machines, ready reckoners and calculating machines. But it is still important that all calculations should be accurate to whatever degree of approximation is being used.

One of the easiest and most widespread ways of making an approximate calculation is to use a slide rule. Time is saved this way and the pupils are forced to make rough approximations (see p. 88) in order to place the decimal point. Their use raises an interesting topic for discussion, namely the degree of accuracy which can be achieved with slide rules of different lengths and on different parts of the scales.

Notes. (1) The statements that an angle is 17·50, that it is 17·5, and that it is 17½ degrees do not mean the same thing. The first statement, by convention, implies 4-figure accuracy—a result between 17·495 and 17·505; the second, 3-figure accuracy—a result between 17·45 and 17·55. The third statement is not so clear in its implication; it means we cannot tell whether the angle is nearer 17 or 18; if given as an answer to a measurement it might mean between $17\frac{1}{4}$ and $17\frac{3}{4}$: this can be put briefly as $17\frac{1}{2}$ within $\frac{1}{4}$.

---

* See also p. 72.

(2) A bad way of approximating to a small quantity is to find (approximately) two large quantities of which it is the difference; the weight of the captain's watch is not well found by estimating the ship's displacement before and after it fell overboard. An example of this is the evaluation of $\frac{1}{9\cdot84}-\frac{1}{9\cdot85}$ using 4-figure reciprocal tables:

$$\frac{1}{9\cdot84}-\frac{1}{9\cdot85}\simeq 0\cdot1016-0\cdot1015=0\cdot0001.$$

Here, although we worked to 4 significant figures, we have only got one significant figure in the result, and even this is not reliable.

We could, however, improve the accuracy while still using 4-figure tables by the following working:

$$\begin{aligned}\frac{1}{9\cdot84}-\frac{1}{9\cdot85}&=\frac{9\cdot85-9\cdot84}{9\cdot84\times9\cdot85},\\ &=\frac{0\cdot01}{9\cdot84\times9\cdot85},\\ &=0\cdot000\ 103.\end{aligned}$$

9·84	0·9930
9·85	0·9934
Den.	1·9864
0·01	$\bar{2}\cdot0000$
Den.	1·9864
0·000 1031	$\bar{4}\cdot0136$

In this way we have retained 4 significant figures in our result but the last figure is uncertain. On the other hand, the difference can be found directly by increasing the accuracy of the approximations using long division to express the reciprocals as decimals:

$$\begin{aligned}\frac{1}{9\cdot84}-\frac{1}{9\cdot85}&=0\cdot101\ 6260-0\cdot101\ 5228,\\ &=0\cdot000\ 1032;\end{aligned}$$

but here we have had to work to 7 significant figures in order to obtain the same order of accuracy as in the previous working, the last figure again being uncertain.

(3) British Standards recommend the use of the sign $\simeq$ for 'is approximately equal to'.* A connecting sign of this sort applies strictly to the two expressions connected by it,† so that we should write:

$$\pi\,.\,7^2\simeq\frac{22}{7}\times7^2=154.$$

or

$$\begin{aligned}\pi\,.\,7^2&\simeq\frac{22}{7}\times7^2,\\ &=154,\end{aligned}$$

from which we deduce that $\pi\,.\,7^2\simeq154$. It may be helpful, in this example, to read the '=' sign as 'which is equal to'.

(4) It is a useful working principle that when an approximation is divided

---

* British Standard 1991: Part I: 1954, *Letter symbols, signs and abbreviations*, p. 14.

† See T. W. Chaundy, P. R. Barrett, Charles Batey, *The printing of mathematics*. (O.U.P.) 1954, p. 38.

by n (>1) the error is also divided by n. This is the basis, for example, of the method of finding the circumference of a cylinder by measuring the length of several turns of string. On the other hand multiplication of a measurement by n exaggerates the error. If a metre rule with an error of 0·01 cm were used to measure a kilometre, an error of the order of 10 cm might be expected. This principle also has applications to errors arising in the course of calculations. For example if $\frac{2}{3}$ of 0·5186 is calculated by first dividing by 3 to four significant figures and then multiplying by 2, the last figure of the answer has an error of 1. This error could have been avoided by first multiplying and then dividing. It can also be avoided by working to one more place than is required in the answer.

Another aspect of the use of approximations to which some attention should be given is the value of delaying the substitution of an approximate value. For example in calculating $3{\cdot}42\pi + 1{\cdot}85\pi$ it is simpler to write it in the form $(3{\cdot}42 + 1{\cdot}85)\pi$ before substituting a numerical value. Such work may well be taken with the use of brackets in algebra.

CHAPTER 14

INTRODUCTION TO STATISTICS

IF the members of a class measure the same quantity and the different answers are collected, it is natural to ask whether it is possible to derive an estimate more reliable than the individual measurements. Such work, which can begin quite early in the arithmetic course, provides an introduction to some of the elementary concepts of statistics, e.g. average or mean, and median.

For example, each pupil measures the length of his shoe and is then asked to measure the length of the classroom floor by a 'heel and toe' method, i.e. by counting the number of shoe-lengths. On one occasion the following measures were obtained and put on the board in order of magnitude: 19′ 4″, 21′ 1″, 21′ 7″, 22′ 1″, ..., 24′ 3″, 24′ 4″, 25′ 5″, 28′ 3″. There were 27 measurements in all. After some discussion the first one and the last two were struck out as being 'obviously wrong' and the average of the remaining 24 was found to be 23′ 1″. Then and only then, a tape measure was discovered and this yielded a measurement of 23′ 4″. The calculation of the average is lengthy and methods of shortening it will be welcome. A typical suggestion is to calculate the average of the excess of each measurement above 21 ft. If the measurements are arranged in order of magnitude it is natural to introduce the median (the central value or, if the number of measurements is even, the average of the two middle ones) as a more easily found central number. This in turn may suggest the advantage of taking a trial average near the median and calculating the average of the deviations by a process of cancelling as in bridge scoring.

$$\begin{array}{cc} - & + \\ \hline \cancel{2} & \cancel{12} \\ \cancel{8} & \cancel{7} \\ \cancel{6} & \cancel{3} \\ \cancel{0} & 2 \\ \cancel{3} & 2 \\ \cancel{4} & \cancel{5} \\ \cancel{2} & \cancel{7} \\ \cancel{1} & \cancel{1} \\ \cancel{6} & \\ \cancel{3} & \\ \hline & +4 \end{array} \qquad \text{Average} = \text{trial average} + \frac{4}{18}.$$

[Zero deviations are recorded to enable the total number of entries to be checked.]

Similar exercises which have been found suitable in the first year of a secondary school are as follows:

(1) When introducing the metric system let groups draw lines of various lengths (1, 2, . . ., 6, 7 in.), measure them in cm, and estimate the length of 1 in. in cm. Collect results and calculate the average.

(2) Distribute maps of a region such as Wales duplicated on squared paper by means of the roller apparatus used in the geography department. Estimate the area of Wales by counting squares and determine the average and/or median.

(3) Let each pupil determine by means of measuring tapes in the playground the distance he walks in, say, 100 normal paces. Then determine an easily defined local distance, e.g. length of a football pitch, distance from school gate to station, etc. Determine the average. When asked for the 'right answer' look up the figures obtained in previous years, and perhaps consider how to find an average of averages. (The number of measurements as well as the average must, of course, be recorded each year.)

A second use of the average (or mean) and of the median as a convenient representative of a group of numbers must be introduced at some stage. This is the purpose of such numbers as average age, average mark etc., and this use occurs particularly when it is desired to compare groups of numbers. Other examples are average rainfall, average temperature, batting and bowling averages. Metereological averages can be related to local observations, and a recent copy of Wisden's is an excellent source of exercises on cricket averages. Statistical surveys, such as collecting travel times to and from school, stimulate interest and raise new problems in handling results. This is the place in which to introduce the ideas of class interval, frequency, and the graphical display of statistics. If the following recording system is used, the diagram itself is a rough histogram:

Marks obtained in a test						
0– 9	𝍸					
10–19	𝍸	𝍸	|||			
20–29	𝍸	𝍸	𝍸	𝍸	||	
30–39	𝍸	𝍸	𝍸	𝍸	𝍸	||
40–49	𝍸	𝍸	𝍸	𝍸	𝍸	
50–59	𝍸	𝍸	𝍸	||		
60–69	𝍸	||				

Throughout the work in statistics, diagrams of all kinds should be in constant use. Not only should they be drawn by the pupils but specimens of different types should be collected.

Artificial questions designed to correct exaggerated or mistaken ideas about the information conveyed by a mean value have their place. For example:

(i) In a test two groups of 10 pupils obtained the following marks out of 10:
Group A. 1, 4, 5, 5, 5, 5, 6, 6, 6, 10
Group B. Each pupil had 7 marks.
Group B has the better average but Group A contains both the lowest and the highest mark.

(ii) *A* has played 10 innings and has an average of 20. *B* has played 4 innings and has an average of 19. In the next match *A* scores 31 and *B* scores 30. What are now their averages? It surprises the pupils to find that *B*'s average is higher than *A*'s.

There will be opportunities for the discussion of the meaning of the statistics and the interpretation of the results of the calculations. For example, the accuracy to which an average is calculated depends upon the purpose to which it is to be put and upon the accuracy of the data.

A full discussion of the advantages of the median and mean cannot be made until the concept of standard error has been developed. This is because the most cogent reason for the more frequent use of the mean is that its standard error is less than that of other measures of central tendency. However, some discussion of the reason for having these two representative numbers is almost certain to arise from the fact that a choice has to be made between them; and the discussion will range wider still if the mode (see below) is mentioned, as it deserves to be. The *median* is sometimes quicker to calculate than the mean; for example, a form order of marks is required in many schools and, if this is available, the median can be found immediately. When there are a number of extreme values which would influence the mean, the median is likely to give a more typical value: and a similar situation arises when a distribution is markedly skew. Another case where the median is sometimes preferable is when a distribution is tabulated with one or both ends left open, e.g.

Number of subjects taken at O level of the G.C.E.	3 or less	4	5	6	7	8 or more
Number of candidates	20	12	23	35	21	17

Here the number of candidates has been given for each of the more central numbers of subjects and the median can be confidently stated to be 6; but in the distribution of marks above, while it is quite easy to discover that the median lies within the range 30–39 marks, its estimation to the nearest mark is an interesting exercise in interpolation. The mean has the advantage to pupils that it is familiar, and it is more easily defined with clarity. If $f_1, f_2, \ldots$ are the frequencies of the measurements $x_1, x_2, \ldots$, the mean is calculated from the formula $\frac{\Sigma f_1 x_1}{\Sigma f_1}$. There is no such simple formula for the median. From here it is easy to go on to point out that this makes the mean easier to deal with in theoretical work and that when calculating machines are available the computation involved in calculating the mean is not nearly so laborious as it is with pen and paper. The *mode* is the value of the quantity measured which occurs most frequently, e.g. 6 in the table above; but when the measurements are collected in classes, as in the rough frequency histogram above, it is better to refer to the 'modal class'; there the modal class is 30–39 marks.

At a later stage in the course, the question of the reliability of averages may be raised with profit. The early work on averages may have led to an uncritical acceptance of such statistics: a class will readily see that the average of their individual estimates of a length, of an area, of π, or of any measurement they may make, is likely to be more reliable than a single result. What may not be clear to them is the fact that the average itself would almost certainly have been different if the measurements had been performed by another class. At some stage in a pupil's education, he should be taught 'how to talk back to a statistic' (Darrell Huff. *How to lie with statistics*); at present many 'educated' people have very little idea of how to examine critically statistical evidence of even an elementary nature.

The pupils will probably have asked themselves such questions as, 'What is the average height of men (or women) in this country?' They may even quote a figure with disarming certainty: if so, here is an opportunity to sow doubt on the authority of the source. Anyway, such a problem, posed by teacher or pupil, can give rise to the problems of deciding what set of individuals to measure or question. This will be most effectively illustrated if the pupils undertake a survey themselves. Inside the school they may be interested in finding out the average number of bus journeys made per week or the average time spent on homework in the same period. Heights, weights, ages of pupils or parents, number of brothers and sisters, books read in a year, pocket money, size of shoe, rates of reading, etc. can be treated in the same way. Outside school, much will depend on the locality and on opportunities of individual pupils. The average amount spent by customers in a shop, consumption of electricity, rateable values, traffic densities, and the weight of a standing crop are examples of surveys which present problems of making the necessary observations.

When individuals or groups investigating the same population obtain different averages, doubt is cast on the reliability of the results. It is important to distinguish between the variation due to making observations which are not representative and the variation due to chance. The importance of the process of selection has been mentioned above: the variation due to chance needs more advanced mathematics for a full treatment and it is intended to provide a discussion of this matter in a future Report on the Teaching of Statistics in Schools. However, an example can give pupils some insight into random fluctuations of a mean. A simple classroom device is a pack of plain cards, numbered consecutively from 1 to 100, although the pupils are not told this yet. On each of a number of occasions, ten cards are drawn at random (but not replaced until all 10 have been drawn) the average is recorded, the cards replaced, and the pack shuffled. With several identical packs, the pupils can work in small groups and 50 averages obtained quite quickly. From such a pack of cards, fifty averages might be distributed as in the figure. We note that:

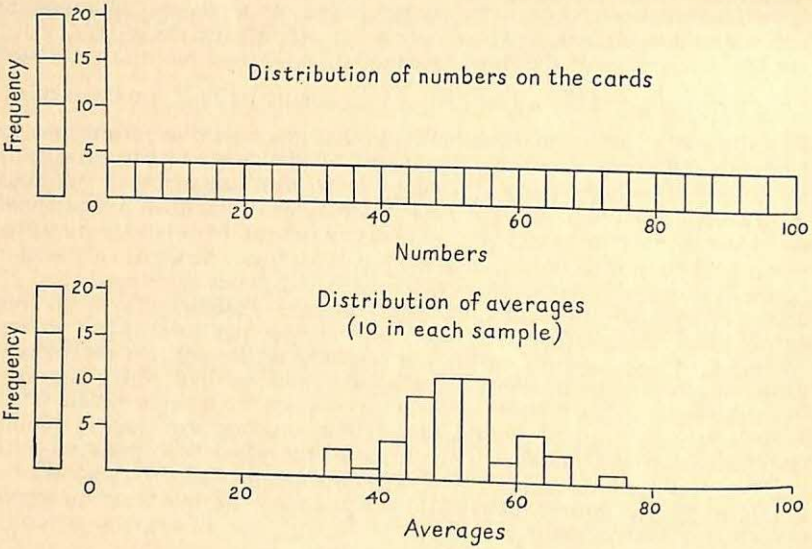


FIG. 14.1

(i) the averages appear to be clustered about the mean of the numbers in the pack;
(ii) the distribution is roughly symmetrical;
(iii) the averages are far less spread out than the numbers on the pack.

Since averages of groups drawn from the same population vary, it can be seen that we cannot know what reliance to place on an average unless we are given some indication of how much variation may reasonably be expected. Statements about averages in the press can be examined in the light of this knowledge but teachers should not allow the discussion to range outside their own knowledge of statistics.

CHAPTER 15

TRANSITION FROM THE PRIMARY SCHOOL

WORK attempted at the primary stage is changing rapidly both in content and in approach. There is a wide range of attitude, attainment and ability to be considered in entrants to all types of secondary school. It is however generally true to say that for the majority instruction in arithmetic has loomed large at early stages. A great many entrants have covered a good deal of the groundwork of computation and spent time practising rules often without acquiring any real mastery of the material involved, though perhaps gaining considerable facility in reckoning.

Since there is so much variation in the background of pupils entering the grammar school, some teachers have begun the mathematics course with prolonged revision. This is a mistake. It is important that a pupil entering a secondary school should get as much thrill as possible out of this new stage of his education. A mathematics course must certainly include opportunities for revision but, in arithmetic particularly, all the work is, or can be, revised when applications of it are made later on in the course. If more frequent revision is needed, this can be done in brief spells interspersed in new work.

In some cases a pupil has failed to assimilate (or even missed altogether) matter far behind the apparent growing point of his knowledge. Then a special approach is needed which may be called remedial treatment; this has been discussed at length in Chapter 4 of the Association's Report, *Mathematics in secondary modern schools*. The problem is one which arises more frequently in modern schools but every mathematics teacher in a grammar school should be ready to undertake this work whenever it is needed. The reader is also referred to the same Report for suggestions about assessing the outcome of mathematical education before the secondary stage. The relevant sections are: 2·0, 2·01 and 2·4 to 2·43.

It is important that the teacher of an entrant class should acquire as much information as possible about the attainments and attitudes of his pupils. Some of this may be obtained from primary school record cards, or from contact with the primary school teachers and much can be gathered from conversation with the pupils. However, children often talk about an idea without having developed a correct concept of it. This happens in infancy when children use number words without understanding their meaning and it goes on, unless the teacher is very careful, right through the school course of mathematics. In the field of mensuration, particularly, children often arrive at the secondary school with only partially developed concepts of area and volume. Some suggestions for work to help such pupils are given in Chapter 9. It is not suggested that a teacher should refrain from making progress in a particular field because the relevant concepts are undeveloped. Rather, the concepts are developed when the pupil acquires experience in that field; but at present work designed to help pupils in this way (as opposed to simply teaching computational techniques) is not as common as it should be.

'Stage A' geometry was described in the Association's Report, *The teaching of geometry in schools* (1923). The work mentioned in the previous paragraph might well be described as 'Stage A arithmetic'. Its main purpose is not so much to communicate knowledge as to help our pupils to understand the concepts with which arithmetic deals.

CHAPTER 16

SOME SUGGESTIONS ON CLASSROOM PROCEDURE

The first stages in teaching a new topic are particularly important since first impressions endure and may permanently influence a pupil's thinking. It is generally recognised that simple examples are best, chosen not only to ensure successful working out by the pupil, but more importantly so that the mathematical thinking is not obscured by the intricacies of the calculation. Special cases may be misleading in that they do not give the pupil a reasonable view of the topic. It is for this reason that in the section on graphs in this report we suggest beginning with a few curved graphs so that the general technique of plotting points and joining them with a smooth curve is met before the special technique of using a ruler when the graph happens to be a straight line. Simple proportion, again, is only one kind of functional relationship and should be put in perspective early by giving examples of other kinds, even though it may not be possible to do much with them. The examples should gradually become more complex and too much stress should not be laid on those which start from exact data or 'come out' exactly, lest quite a wrong idea is obtained of the way in which mathematics is applied in real life.

The teacher has, broadly speaking, two lines of approach. The first is traditional and familiar: it is teacher-centred and relies on exposition and examples worked on the blackboard, but the pupil learns largely by listening. In the second the emphasis is on the pupil's activity: the teacher's task is to contrive the appropriate stimulus to learning, sometimes by posing a challenge or problem and guiding the pupil to discover a solution, at other times by suggesting an activity, e.g. the use of a calculating machine, leading to the discovery of a generalisation. This method of 'guided learning,' as it has been called, has always been used by some teachers in presenting any subject and is now becoming more acceptable in mathematics through the pioneering work of John Perry and Sir Percy Nunn, with its successful implementation in our generation in infant and junior schools. Essentially one begins with a problem (The recipe says 1¼ lb of sugar to 1 lb of fruit: we have 3½ lb of fruit. . . : what can we do to answer this? . . . One refrigerator has these dimensions . . . , another these . . .: how can we compare the two? . . .) and uses the natural curiosity and intelligence of the children to evolve a solution. In so doing, it is important for them to be ahead of the teacher in their thinking and reckoning (as in doing riders in geometry when they draw their own diagrams before any diagram is drawn on the board by the teacher). A more advanced instance of this approach is that of F. C. Boon who used to encourage pupils to discover how a table of logarithms worked by getting them to compare the results of very simple operations with entries in the table. After some guided problem-solving of this kind, a standard pattern of thinking and operating will emerge and this can be summarised on the blackboard and noted by the class: after which elaborations may be developed, but fundamentally children are thinking and doing for themselves.

They can for instance discover for themselves that the distance round the boundary of a circle is about three times its diameter. This is likely to lead someone to ask how the ratio can be found more accurately. The teacher can invite suggestions and possibly contribute one of his own, and set different groups to follow them up. During the resulting activity the pupils are compelled to think about the relationship of diameter to circumference. They have time to take in its implications and to understand how it fits into their existing knowledge. When the results of the group activities are pooled the whole class is likely to have a clearer understanding of the significance of π than if it had been given a definition and a set of exercises to work. Such a method, of course, takes time and is often slower than the old-fashioned talk-and-chalk, but it can be enormously rewarding for teacher and pupil if real understanding is achieved. No doubt as calculating machines and mathematical laboratories become more common in schools pupils will have more opportunities of learning for themselves how things work as if they were real craftsmen or researchers. And then it may be necessary for teachers to guard against the danger of proceeding too slowly. Obviously the pupil has not sufficient time to retrace the researches of the ages—and the teacher's role must be to shorten the process by suggestion and systematisation. His task is not to expound to a passive class, but to let a new principle or generalisation crystallise as quickly as possible from the children's own thinking. An excellent illustration of this method in action will be found on p. 46 of *Mathematics Teaching* No. 23, Summer 1963.

Eventually the general principles of the subject and the rules of thumb must emerge but here, perhaps more than anywhere else, the teacher will do well to remember the old motto 'more haste, less speed'. Premature introduction of a rule of thumb usually leads to subsequent breakdowns of memory, and tedious bouts of revision. It is better to delay the final generalisation until it is almost spontaneously suggested by the pupils themselves. It is vitally important to give a clear justification for any rule-of-thumb; some pupils will understand it, a few will remember it, but all should realise that there is a justification and that it is logic and not magic.

On the other hand, it is most important that this final crystallisation of a fluid method into a hard-and-fast rule should be well defined. The pupil must know and use such rules-of-thumb, and the habit of using them must be developed. Mathematics essentially aims at an economy of thought, and the more operations a pupil can perform without having to stop and think, the more he will learn about the subject. 'Civilisation,' as Whitehead says, 'advances by extending the number of important operations we can perform without thinking about them.'

Ultimately the basic facts and rules of arithmetic must be known so thoroughly that they can be recalled without conscious effort. This takes time and patience, but one method by which the process can be helped is the setting of a few quick questions at the beginning of the arithmetic lessons for as long as may be necessary. These questions should be easy so that most of the class get them right, and not too numerous so that they do not encroach too much upon the lesson. Their aim is simply to recall repeatedly the fact or process it is desired to learn. It may be the number of square yards in an acre, or the decimal equivalents of simple fractions such as $\frac{3}{8}$, or the division of fractions, e.g. $\frac{3}{4} \div \frac{2}{3}$. There is no need to restrict ourselves to arithmetic, and the question just quoted might well be followed by something like $\frac{3}{a} \div \frac{2}{b}$, or $\frac{3}{x} \div \frac{2}{x}$. Apart from the value of the repetition, these quick questions can be made into a kind of game, and then they can give the pupil the impression that the teacher, for his part, is doing all he can to help him to learn. This is so much better than throwing the whole responsibility on the pupil with the remark, 'Learn this for homework. I will test you tomorrow.'

But, of course, the staple diet of the classroom consists of longer written exercises. These are needed to give experience and to consolidate the grasp of a new idea. Care should be taken with the choice of examples and there are several pitfalls. While consolidation requires the working of a number of examples, this is sometimes carried to excess. On the one hand, success with a number of similar examples gives a pupil confidence; on the other hand, repetitive exercises will tend to associate a new process with a particular type of example and will become boring. The teacher therefore needs to draw a balance with each particular class between examples which give confidence and those which will challenge and stimulate.

When examples are set from a text book, care should be taken to check that they are well graded and suitable for the class. Examples which are not taken from the text book have a freshness about them which the printed ones lack. Every effort should be made to bring the teaching in the classroom into close relation with the life of the child outside it and information brought by pupils on their own initiative should be used if possible. Some may have parents or friends with specialised knowledge and can bring in data of particular interest. Anything with a numerical aspect which interests the class is a valid basis for a piece of arithmetic. The best exercises are those which arise out of questions asked by the pupils themselves and occasionally it may be worthwhile inviting them to make up their own exercises, some of which may be set to the rest of the class. But there is one difficulty in the use of real-life examples—the computation is often so heavy that it distracts attention from the principles involved. One way of dealing with this difficulty without losing too much interest is to make a suitable approximation. For example, a young boy brought to a mathematics lesson the information that Zatopek ran 10 000 metres (6 miles 376 yards) in 29 min 2·6 sec. This led to, and gave added interest to, the simple question, 'How fast is a man running when he covers 6 miles in 30 minutes?'

Examples of the type we are considering here must be written down, and almost immediately the teacher is confronted with a dilemma: should he require the accurate use of words and symbols from the start, so as to lay the foundations of good habits and prevent bad ones from being acquired, or will such a course tend to make the work seem artificial to the pupil?

Feeling for style and accuracy of expression, and for form in the presentation of work, are of gradual growth. Children use a small vocabulary for ordinary talk, not only because they have a limited stock of words but because of their limited experience and their ignorance of the finer shades of meaning. They use words loosely and their thought is unorganised; and this is all quite natural. It is the task of the teacher, in any subject, to show them how to think more clearly and how to express their thoughts on paper.

In teaching mathematics there is a danger of demanding too much in this way at an early stage. Accuracy of expression may be always desirable, but only good judgment can determine how much should be required at any given time. To decide this question the teacher must keep his hand on the pulse of the class; interest will flag when there is too great an insistence on what is, to the immature mind, merely detail.

A good style in the presentation of mathematical work is a valuable aid to accuracy and leads to economy of thought and time. Good style consists in telling the truth simply and clearly. It is the product of clear thought and thus often the index of ability; but also it can be taught. Bad style is the result sometimes of failure to understand; but more often of indolence or lack of teaching. It is a trial to the reader's patience and a handicap to progress.

The necessity for clear expression is not apparent to the beginner; nor can he be left to learn by experience. He must be shown on the blackboard what good style means and must be encouraged to acquire good habits. Any effort he makes towards clear expression will bring its reward of increased accuracy and comprehension, enabling him to tackle larger scale work as time goes on.

A well-written piece of arithmetic is more easily checked by reading through than by doing the work again. It consists normally of sentences, accompanied by 'sums' or 'working'. The 'working' is tabulated, but to the sentences the usual rules of grammar apply. A good test is to read the sentences aloud. The '=' sign is a verb, and '∴' is a conjunction. The working must never be hitched on to the sentences, e.g.,

$$\text{`The number of boys per bus} = \frac{230}{9} = \frac{9\,)\,230}{25{\cdot}56},$$

nor yet relegated to a 'rubbish-heap' margin; sometimes it should be placed in the middle of the page, more often at the side, opposite the sentences to which it is relevant. It should consist normally of operations on numbers, the units being dealt with in the course of the sentences; and the numbers should be written full size and arranged neatly in columns.

Mis-use of the '=' sign should not be tolerated: e.g. '3 horses = £80', '50 per cent = £26', or (in a question about a doctor's practice) 'losing patients = 20 per cent'. A scale should be stated as '1 inch repr. 2 miles' or '1 in. : 2 miles'. If x is used, it should represent a number, not a sum of money or other quantity. A solution should not begin 'Let x = time', nor finish '∴ $x = \frac{2}{5}$ h = 24 min'. The answer should be written as a sentence: 'Time taken = 24 min.' The writing of 'Answer = 24 min' should be discouraged.

Good style is more easily caught than taught, and the teacher's own style on the blackboard will have more influence than precept, reward or punishment. But direct teaching is necessary as well. Children can be taught at an early stage to use their limited vocabulary effectively and to build up the habit of clear expression. The writing of such simple sentences as 'Cost = ...', 'Area = ...', 'Time taken by Smith = ...' makes no great demand on their patience, but goes far to give them the idea that what they put on paper should have meaning. The practical-minded pupil who says 'It gets the answer right and that's all that matters' can be induced to admit that 'we ought to say what we're talking about'.

This is not to say that there is no place for paper jottings. There are many occasions in everyday life when the backs of envelopes can be usefully pressed into service, and in the classroom too it is not every piece of work that should be fully written out. Sometimes the teacher, perhaps when explaining something new, may want the class to share in the work by doing some of it for themselves on rough paper. Some work again is best arranged in tabular form.

The indifferent style which is sometimes adopted by pupils may be due to their keenness; their thoughts are far ahead of their pens. But it may also be due to their being given work which is too easy or too difficult; in the first case the writing of logical statements is apparently superfluous, in the second insistence on good style may be the last straw which makes the whole task seem too formidable to be tackled.

A caution must be added: teachers, usually those with some experience, may require perfection of style too soon. This may have the effect of substituting the mental operations of the teacher for those of the pupil. A mean has to be observed between the toleration of slipshod work and a requirement of such precision as can only be satisfied by dictation. To each age corresponds its own vocabulary and its own standard of exactness of thought and expression. Every traveller on the continent knows how a limited vocabulary cramps his conversational style, and anyone who explores unfamiliar mathematical ground knows that understanding of a vague kind must be followed by a further effort before the new idea can be expressed clearly on paper.

The difficulty of clear expression must not be under-rated, but neither must its value. The pupil may be right when he regards the good presentation of the work as an additional task, but he must be brought to see that the effort is worthwhile.

One of the attractions of mathematics is the relative certainty of the conclusions reached in it. Pupils are always pleased to tick their answers right or to see the teacher's ticks, and the greater the proportion of answers right, the greater will be the enthusiasm of a class for arithmetic—provided that the questions set are not so easy that the pupils think them a waste of time. Checking is an important aid to accuracy and is not a part of teaching that can be ignored with impunity.

First, it should be stressed that checking without the right attitude to it is a waste of time. Pupils making good progress in mathematics want to get their work correct and should see that checking is a means to this end. A check that is done because the teacher requires it, and not as a help to getting the answer right, is useless. The object should be to encourage the habit of checking. There are two questions that pupils should ask themselves on completing a piece of arithmetic: 'Is it a reasonable answer?', 'Is it right?' Ways of trying to answer these questions are discussed in the paragraphs below.

Perhaps the most important thing for a pupil to check is that he has done what he meant to do. This may be encouraged by giving answers in the form: 'The volume is 36 cu. in.' or 'The surface area is $5\frac{1}{2}$ sq. yd,' rather than: 'Answer 36 cu. in.' or 'Ans. $5\frac{1}{2}$ sq. yd'. Next he should ask himself, 'Is this a reasonable answer?' If, for instance, a car going from London to Winchester has an average speed of 325 miles per hour, or a pound of sugar costs $\frac{1}{2}$d., he should see that these are not consistent with everyday experience. In this respect text-books should not be too out of date, although it is interesting for the pupils to compare pre-war prices with those of the present. It will be seen that a written check cannot be expected in this comparison with everyday life, but some answers to the question, 'Is it reasonable?' can be written down. For instance (see Chapter 12) logarithms are used to evaluate an expression and the answer 29·1 is obtained. A rough approximation to check the position of the decimal point can be written down as follows:

$$\frac{18{\cdot}7 \times 15{\cdot}8}{216 \times 0{\cdot}047} \simeq \frac{20 \times 15}{200 \times 0{\cdot}05},$$

which is easily evaluated. In this particular case, the lower streams will find the idea of approximation difficult and it is well to bear in mind the principle that any check should be simpler to the pupils than the calculation that is being checked. Pupils who can follow a routine method, but find it hard to think for themselves do not take easily to rough approximations. It can be useful to suggest that approximations to one significant figure are sufficient.

In considering the ways of answering the question, 'Is it right?' it is well to think what mathematicians and pupils do when they really want to be sure of getting a correct result. One of the most frequent ways is to compare the answer with someone else's answer to the same question. This is what is often done in adult life and what pupils do with their homework or prep (and sometimes in class). Again, this is what the teacher and pupils do when an answer book is used. If no one else's answer is available, the best method of checking is to work the question again by a different method. This is what is done in checking addition by adding up and down a column of figures. Another method of checking an answer is to use that answer to deduce something that is already known, for instance, the errors in the subtraction

$$\begin{array}{r} \bar{1}\cdot3043 \\ \underline{\bar{2}\cdot6859} \\ 1\cdot6284 \end{array}$$

will easily be detected by addition. Similarly division may be checked by multiplication. An advantage of checking by a different method is that it eliminates the very real risk of making the same mistake twice. Readings from tables should be checked—comparing what has been written with the entries in the tables to detect miscopying.

Practical work introduces calculations to which the answers are not verifiable by looking at the back of the book. They can be used to introduce the idea of checking by the consistency of several results. The area of a triangle, for example, can be found by measuring a side and an altitude in three ways, or the different calculations of a height or distance made by the members of a class can be compared.

Inaccuracy often is the result of untidy work and slovenly figures, and finding a mistake can be made much more difficult if part of the calculation is crammed into a small space. The teacher's own attitude to tidiness and checking will make an important contribution to the accuracy of the pupils' work. If marks are given for quantity, it will generally pay to do a large number of questions without checking and with little regard for tidiness. But above all, the teacher should bear in mind that the educational process involved in checking is only complete when the pupil habitually applies checks that will make him fairly sure that his work is correct for his own satisfaction.

But even if this ideal is realised there will still be mistakes and we have to consider what to do then. Before deciding upon the traditional plan of setting 'corrections' the teacher should have a clear idea of the purpose he is trying to serve. Mistakes in arithmetic may be divided into two classes: those in which the pupil has not applied some principle correctly and those in which some mechanical mistake has been made, frequently inaccuracy in the use of tables or in simple addition or subtraction. From the work alone, it is sometimes difficult to decide into which class a particular error falls: it is the pupil's understanding of the process which is the determining factor. In the first case corrections or, better still, similar examples may be set. In the second case corrections may produce the opposite effect to that which is desired. If a child is bored with the work, he will not take such care over it and further mistakes will probably follow. Too many corrections slow down the pace, and interest in the subject may be lost.

Before corrections can usefully be done it is necessary for the pupil to find out or be told where he has gone wrong. In a large class this may take a long time, and it may be useful to let some of the better pupils act as group leaders and give help to the members of their own group. The ideal is to give the least amount of help which will enable the pupil to put right what is wrong. It may be 'That's wrong. Try and find the mistake yourself and put it right', or it may be 'The mistake is here: go and correct it', or it may be a series of questions which end with the pupil saying, 'I see. How silly of me.' This ties up with checking since, if the helper does nothing but ask questions, it may be pointed out that the pupil could have found the mistake by himself, if only he had thought of the questions.

When new pupils in a secondary school make mistakes it is tempting to lay the blame on the methods taught in their old school, where these differ from those favoured in the new school. But before a teacher attempts to alter the habit either of work arrangement or of a thought process of a pupil, he should consider carefully his reasons for doing so. Uniformity of method within the class makes for ease in class teaching, but this is not, in itself, sufficient reason for attempting to change the habits of any pupils whose methods may differ from those of the majority. A habit, once formed, may be difficult to alter, and can only be altered with the co-operation of the pupil. If the method employed is giving good results no attempt should be made to change it unless the teacher is sure that it will ultimately prove a hindrance to progress in later work.

The more able the pupil the more flexible he will be. When the use of logarithms is being taught, and the need to subtract numbers with negative characteristics arises, the teacher must be sure which method of subtraction is being used by each pupil. At this stage it is not unknown for a child who has been taught decomposition, voluntarily to change to equal additions, or complementary addition, and to persist with the new method in all future work. No attempt should be made by the teacher to compel such a change against the will of the pupil, or confusion will certainly result.

It is desirable, whenever possible, to teach in the first instance the arrangement of work which will be acceptable in all future work. Habits which tend to economy of thought should be introduced when the pupil appears ready to receive them, and should be encouraged but not enforced.

APPENDIX ON UNITS

THE *Weights and Measures Act* 1963 defines the yard as 0·9144 metre exactly and the pound (mass) as 0·453 592 37 kilogramme exactly. The metre, kilogramme, litre, ampere, ohm, volt and watt are defined internationally and this country adopts the international definitions in force from time to time.* The Act defines the gallon as 'the space occupied by 10 pounds weight of distilled water of density 0·998 859 gramme per millilitre weighed in air of density 0·001 217 gramme per millilitre against weights of density 8·136 grammes per millilitre'.

The units of measurement of length, area, volume, capacity, mass and weight which are permitted to be used in trade are set out in Schedule 1 of the Act. They are listed below:

Imperial units	*Metric units*
Measurement of length	
Mile	Kilometre
Furlong	Metre
Chain	Decimetre
Yard	Centimetre
Foot	Millimetre
Inch	
Measurement of area	
Square mile	Hectare
Acre	Dekare
Rood	Are
Square yard	Square metre
Square foot	Square decimetre
Square inch	Square centimetre
	Square millimetre
Measurement of volume	
Cubic yard	Cubic metre
Cubic foot	Cubic decimetre
Cubic inch	Cubic centimetre
Measurement of capacity	
Gallon	Hectolitre
Quart	Litre
Pint	Decilitre
Gill	Centilitre
Fluid ounce	Millilitre
Bushel	
Peck	
Fluid drachm	
Minim	

---

* The current definition of the metre in April 1964 was 1 650 763·73 wave lengths in vacuo of the orange-red line of krypton-86. In October 1964 the second was redefined as 9 192 631 770 oscillations of the caesium-133 atom.

Measurement of mass or weight

Ton
Hundredweight
Cental
Quarter
Stone
Pound
Ounce
Dram
Grain
Ounce troy
Pennyweight
Ounce apothecaries
Drachm
Scruple

Metric ton
Quintal
Kilogramme
Hectogramme
Gramme
Carat (metric)
Milligramme

Measurement of electricity

Ampere
Ohm
Volt
Watt
Kilowatt
Megawatt

The bushel, peck and pennyweight cease to be lawful units five years after the Act comes into force.

The fluid drachm, minim, the ounce apothecaries, drachm and scruple will cease to be lawful units at some later date.

The carat (metric), the ounce troy, the ounce apothecaries, drachm, scruple, fluid drachm and minim are restricted in their use.

Amongst the lesser known units in general use,

1 rood = 1210 square yards,
1 are = 100 square metres,
1 gill = $\frac{1}{4}$ pint,
1 fluid ounce = $\frac{1}{20}$ pint,
1 cental = 100 pounds,
1 dram = $\frac{1}{16}$ ounce,
1 grain = $\frac{1}{7000}$ pound,
1 metric ton = 1000 kilogrammes
1 quintal = 100 kilogrammes
1 carat (metric) = $\frac{1}{5}$ gramme

Schedule 2 of the Act describes the existing United Kingdom Primary Standards and Authorised Copies thereof.

INDEX

References are to pages. Where there are two or more references under one heading, bold type indicates a principal reference.

Abacus, 9
Accuracy, 45, **74–78**
  of expression, 86
  of tables, 72
Addition, 13
  of decimals, 24
  of fractions, 21
Algebra, 52, 56
  laws of, 23
Antilogarithms, 69
Approximations, 74–78
Arrangement of work, 69, 90
Area, 16, **45–52,** 91
Averages, 79–82
Axiomatic approach, 4

Binary scale, 3, 4, **10**
Books, 12

Calculating machines, 3, 4, 10, 24, 71, 81
Cancelling, 23
Capacity, 48, 91
Checks, 88
Circles, 49
Classroom procedure, 84–90
Complex numbers, 7, 8
Compound interest, 41
Compound proportion, 38
Compound units, 15–19
Computation, 14
Cones, 50
Continued fractions, 64
Contracted methods, 1
Corrections, 89
Counting squares, 46, 49
Cylinders, 49

Decimal places, 45, 75
Decimal point, 27
Decimalisation, 18
Decimals, 20–28
Decreases, 31, 39
Degree of accuracy, 15, 45, **74–78**
Depreciation, 41

Dimensions, 52
Direct proportion, 38
Directed numbers, 7, 8
Discount, 30
Division, 13, 14
  compound, 18
  galley method, 10
  of decimals, 26
  of fractions, 22
  scratch method, 10

Electrical units, 92
Estimation, 44
Examples, 86

Four rules, 13, 14
Fractions, 7, **20–28**
Frequency tables, 80
Functionality, 39, 56

Graphs, 34, 43, **54–57,** 59, 65, 70–73
Groups, 4
'Guided learning', 84

Habits, 90
H.C.F., 11

Inaccuracy, 89
Increases, 31, 39
Index numbers, 32
Indices, 65
Inequalities, 3
Integers, 7
Inverse problems, 39–41
Irrational numbers, 7

Laws of algebra, 23
Length, 44, 91
  of curves, 45, 49
L.C.M., 11
Local authority finance, 42
Logarithms, 65–73

Man-hours, 38

Maps, 36, 46
Mass, 92
Matrices, 4
Mean and median, 79–82
Mensuration, 44–53
Metric system, 16, 44, 91
Minus sign, 5
'Modern mathematics', 3–5
Money, 15
Multiplication, 13
  compound, 16–18
  grating method, 9
  of decimals, 25
  of fractions, 21
  tables, 9

Natural numbers, 7
Numbers, 6–12
Numerals, 6

π, 49, 75

Parallelograms, 48
Patterns of numbers, 4, 6
Percentages, 29–32
Place-value, 4, 14, 24
Practice, 16
Primary schools, 83
Prime factors, 11
Prime numbers, 11
Profit and loss, 30, 40
Proportion, 34–38
Pyramids, 50

Rate, 33–43
Rates, 42
Ratio, 33–43
Rational numbers, 7
Ready reckoners, 17, 34, 38
Recurring decimals, 25
Roman numerals, 6
Rough estimates, 27, 88
Rules-of-thumb, 85

Scale drawing, 38
Scales of notation, 4, 10
Sets, 4, 30
Sequences, 4, 11
Significant figures, 45, 75
Simple interest, 40
Slide rule, 4, 38, **72**, 76
Speed, 42
Spheres, 50
Square roots, 58–64
  Newton's method, 60
Squares, 58–64
Standard form, 26, 67
Statistics, 79–82
Style, 86
Subtraction, 13
  of decimals, 24
  of fractions, 21
Symbols, 3–5, 7, 8, 77

Tables, multiplication, 9
  of logarithms, 70
  of squares and square roots, 64
  of weights and measures, 16
Trapeziums, 48
Travel graphs, 57
Triangles, 48
Trigonometry, 38

Unitary method, 35, 37
Units, 15, 16, 44–48, 52, 74, **91–92**

Variation, 39, 56
Volume, 16, **48–52**, 91

Weight, 15, 92
  American units of, 16